The Mathematics of Economic Growth[1]

by

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> An hypothesis is a logical structure. That is, a symbol for which certain rules of representation hold.
>
> Why do I always speak of being compelled by a rule: why not of the fact that I can choose to follow it? For that is equally important.
>
> Wittgenstein

Abstract

Traditionally, economists have considered that mathematics acts as a universal language that lends clarity to theoretical statements. This paper proposes that mathematics does not function as a mere language. Rather, the advocacy of particular theoretical views and the choice of mathematical formalisms go hand-in-hand. The paper explores this issue by investigating the role of mathematics in developments of the theory of economic growth.

I. Introduction

From the late nineteenth century, mathematics has played a vital role in the natural sciences. There has been much borrowing of mathematical formulae by different branches of natural science. Although many economists have argued otherwise, let us grant that economics may have benefitted from such cross-fertilization. The question then is, how have economists benefitted?

The prevailing view of mathematics amongst economists used to be similar to that of logical positivism, the philosophy of science of Comte, Mach and the Vienna Circle. The basic idea was that mathematics permitted economics to be a logical deductive science that yielded conclusions that were empirically testable. Mathematics in this schema, as Samuelson (1952), the expositor of logical positivism in economics perceived, served as a precise, universal language.

By the 1960s, many philosophers of science agreed that the ambitious project of logical positivism was impossible. Not all extra-logical terms could be assigned to objects, the meaning of extra-logical terms suffered from ambiguity, and observations were to some extent theory-bound. Positivism might not offer an unreasonable description of Newtonian science in its reduction of all motion into the principle of least action expressed by means of differential calculus. The reduction of all human action into a few mathematical laws to many minds became unthinkable by the 1930s, with the development of particle physics, involving the identification of numerous elementary particles and irreducible nuclear forces.[2]

Moreover, modern science has seen a veritable revolution in the use of mathematics.[3] Mathematical theory has not only helped solve, but also has generated scientific problems. As Hamilton, whose system of equations marked the beginning of modern dynamical theory, remarked, "while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods" (Hamilton, 1934, p.247). Quantum mechanics originally also went by the name *matrizenmechanik*, after Heisenberg in the 1920s elaborated the Hamiltonian dynamic equations by means of the calculus of noncommutative matrices.[4] In the 1940s, Fermi's statistical simulations on one of the first computers, housed at Los Alamos, led him to invent the Monte Carlo method, which he used to solve many problems.[5] Mathematics in these cases did not act as a language unifying scientific activity. Rather, the construction of particular mathematical tools led scientists to model the world in specific ways.

Many of the difficulties faced by natural scientists have been mathematical in nature. Catching the sense of modern science, the philosopher and historian of science I. Lakatos described "(a) model" as

> a set of initial conditions ... which one

> knows is *bound* to be replaced during the further development of the programme, and one even knows, more or less, how. ... Indeed, if the positive heuristic is clearly spelt out, the difficulties of the programme are mathematical rather than empirical.

(Lakatos, 1978a, p.51). The gain in eliminating mathematical impediments was an empirical one. A program with explanatory power could (i.) generate novel facts, in the sense of explanations of events left unaccounted for by rival theories, (ii.) explain the successes of those rival theories and (iii.) empirically confirm these explanations.

This chapter investigates the effects of using mathematics in developments of the theory of economic growth. It considers the following issues: (1.) Orthodox growth theorists borrowed formalisms from physics to state their theory. To what extent was the analogy between economics and physics thought through? Was the analogy justified? What effect did the mathematical borrowing have on growth economics? (2.) To what extent did the mathematical constructs enforce, rather than merely express, particular views of economists of the growing economy? (3.) Mathematical modelling has an internal logic and imposes certain restrictions on what can be said. Have growth theorists applied their models in ways that conflict with their internal logical? Are the growth models consistent?

This chapter introduces the calculus of nineteenth century physics that economists borrowed and analyses the development of this calculus in Walrasian growth economics in the postwar period (Part 2). Part 3 traces the use of geometry in the development of neoclassical growth theory. Part 4 contrasts Kaldor's informal, growth theory with the formal, neoclassical model. Part 5 looks at new developments in the calculus of Walrasian growth theory, which are intended to encompass the neoclassical and the Kaldorian models. Has this calculus, the chapter asks, merely acted as a universal language, translating what economists already have known or has it given economists the power to think freshly about problems of economic growth? Has the mathematics used in the new growth models created a framework that has too narrowly circumscribed the issues that economists can treat in dealing with growth?

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II. The Mathematical Problem

A. Marginalist Economics and the Energy Integrand[6]

The mathematics of dynamic optimization used in modern economics belongs to the calculus of variations, which was developed to solve certain problems in mechanics in the eighteenth and nineteenth centuries.[7]

One of the earliest problems in the calculus of variations was that of the brachistochrone, the curve of quickest descent. The brachistochrone problem was to determine the curve connecting two points along which a body acted on only by gravity would descend in the shortest time. (The shortest distance between two points in the absence of gravity was of course a straight line. But in the presence of gravity, the particle that moves down a straight line gathers speed relatively slowly. A nonlinear curve that is steeper near the starting point is longer than the straight line, but the particle will traverse the greater part of the curve at greater speed.) Basically, the problem came down to finding the function $x_i=f_i(t)$ that gave the integral I,

(1.) $I\ (x_i,\ \dot{x}_i;\ t_i) = \int_{t}^{t2} L(x_i,\ \dot{x}_i;\ t_i)\,dt, \qquad i = 1 \ldots n,$

in which x_i stood for the position and $\dot{x}$ for the velocity of the particle, an extreme value during the time period t_2-t_1. The solution to this problem involved finding a differential equation that characterized the desired function $x=f(t)$ and then solving the differential equation. The differential, or Euler-Lagrange equation,

(2.) $\delta I = \partial L/\partial x_i - (d/dt)(\partial L/\partial \dot{x}_i) = 0, \qquad t_1 \leq t \leq t_2$

was a second order equation in terms of the position and velocity of the particle. It meant that the quickest descent occurred along the path of the particle at which the integral I became stationary. Along that path, the trade off of the position and the velocity of the particle would be optimal.

Lagrange (1732-1815) developed variational calculus as a way to treat Newton's second law, which equated the force acting on a body to the product of the mass of the body and its acceleration,

(3a.) $F = m\ddot{x}$.

Meanwhile, a new, analytic tradition of mechanics was developing that centered on the notion of the energy scalar rather than the vector of force.

The concept of energy crystallized with the calculus of Hamilton (1805-1865), as a by-product of his study of the variational properties of an optical system. In Hamiltonian mechanics, the Lagrange integrand L was defined as (i.) <u>kinetic energy</u> (T), which was associated with the motion of the particle, <u>minus</u> (ii.) <u>potential energy</u> (V), which was the stored energy of the particle. Thus,

(3b.) $L(x_i, \dot{x}_i; t_i) = T - V$.

The change in potential energy in respect to position gave

(3c.) $-\partial V/\partial x = m\ddot{x}$,

or Newton's second law.[8]

Hamilton's mathematics served to clarify the principle of energy conservation. This principle stated that (i.) total energy (E) was composed of kinetic energy (T) and potential energy (V) and (ii.) total energy in a closed system took on a constant value, that is,

(4.) $$H = \int_{t}^{t2} (T + V)\,dt = \bar{c}.$$

The principle of energy conservation was just like bookkeeping. There was a total amount of energy. The only visible form was kinetic energy, which could be converted into other forms, such as heat or gravitational energy. When the conversion was totally reversible with no loss (as with the frictionless brachistochrone, which involved the conversion of kinetic energy to potential energy as the particle moved from the highest to the lowest position), the form of energy the kinetic energy was converted into was the potential energy function. Energy was a state variable, purely a function of the present values of the system, not on how the system got into that state. If the system were closed, then the force associated with the potential energy was conservative and potential energy was entirely recoverable. Since the quantity kinetic energy plus potential energy was conserved, Hamilton stated that,

> [i]n this expression [equation 4, above], ... the quantity H is independent of time, and does not alter in the passage of the points of the system from one set of position to another"

(Hamilton, 1834, p.250).[9] In contrast, nonconservative sytems were dissipative and necessarily time-dependent.

Hamilton wanted to simplify the solution of the problem of the determination of motion.[10] Instead of the integrating *n* second order equations according to the Euler-Lagrange method, Hamilton integrated 2*n* first order differential equations.

To do this, Hamilton defined the auxiliary variable, momentum, p_i,

(5a.) $p_i \equiv \partial L/\partial \dot{x}_i$,

as the change in the Lagrange function with respect to the velocity of the particle. In Newtonian terms, momentum was mass times velocity,

(5b.) $p=m\dot{x}$.

Kinetic energy was the integral of mass times acceleration. Yhus that[11]

(5c.) $T=\frac{1}{2}m\dot{x}^2=\frac{1}{2}p\dot{x}$.

Given this, Hamilton defined his function as

(6.) $H = -L + \Sigma p_i\dot{x}_i = \overline{c}$, $H = H(x_i, p_i; t)$.

Because (from 3b)

(7a.) $L = T - V$

and (from 5c)

(7b.) $\Sigma p_i\dot{x}_i = 2T$, $L = T - V$,

the Hamiltonian function (equation 6) provided a statement of energy conservation,

(8a.) $H = T + V = \overline{c}$.[12]

Given that

(8b.) $L = -H + \Sigma p_i \dot{x}_i$ (6 rearranged)

Hamilton arrived at the two differential equations,[13]

(9a.) $\partial H/\partial x_i = -\dot{p}_i,$

and

(9b.) $\partial H/\partial p_i = \dot{x}_i.$

These two equations showed the interrrelationship between position, velocity and momenta. Given that H was a constant, one solved for the change in momenta in respect to time, given a change in position (equation 9a) and for velocity, given a change in momenta (9b). The physical system was stationary when $\dot{p}=0$ and $\dot{x}=0$. These two conditions of an extreme path, involving point by point minimization, became known Hamilton's canonical equations. According to Hamilton, these equations achieved the goal of a "general solution of the general problem of dynamics" (1834, p.252).

The energetics movement swept across the various branches of physics. With the introduction of Einstein's theory of relativity at the turn of the century, E. B. Wilson (the physical mathematician and later Samuelson's mentor), saw that "with the very foundation of mechanics sometimes in doubt owing to modern ideas ... the one refuge of many theorists is Hamilton's Principle" (1912, p.415). Modified Hamiltonians played a foundational role in the development of quantum mechanics (1900-1920) and wave mechanics (in the 1920s). Textbooks in physics since have been organized around the conservation of energy. The unity of science movement has seen energy conservation as a universal law. In practice, the Hamiltonian equations were difficult to solve and were rarely used in specific problems.[14]

In economics, the appeal to the mathematics of energy in economics dated back to Jevons (1876) and Walras (1905), who elaborated the theory of static maximization in terms of the so-called "law of energy". Their favorite analogy counterposed the workings of the mechanical lever to economic exchange. This implied the mapping, shown in table 1.

Table 1.

$$W/P = AA'/BB' = AC/BC$$

$$U'(C_a)/U'(C_b) = db/da = b/a,$$

W and P are forces; AA', BB', displacements of the lever; AC, BC, the arms of the lever. a and b stand for too goods.

In other words, just as the relation between the two weights on the lever (W, P) was inversely proportional to displacements from the horizontal (AA', BB'), so the ratio of the marginal utilities (or the prices) of two goods was inversely proportional to their rate of exchange. But the mechanics of the lever, presented by Jevons and Walras, rested on the Newton's third law, that forces always appeared as equal and opposite pairs. This law described a static equilibrium and had nothing to do with motion or the law of energy.[15]

The neoclassical economist I. Fisher (1895/1925) attempted at any length to lay out the purported analogy between maximization in economic statics and energy physics. In a somewhat garbled fashion, Fisher made total utility to be analogous to total energy; total energy less total work, to total utility less total disutility; and the potential, or total work less total energy, to total disutility less total utility.

The early neoclassical economists attempted to apply the mathematics of energy conservation in a static context. Ramsey (1928) solved the problem of optimization in a dynamic context, in which utility was conserved. This marked the beginning of modern neoclassical growth theory.

B. Ramsey (1928)

Ramsey, a teacher of mathematical logic at Cambridge, was, like Keynes, the son of a Cambridge don. A close friend and a former pupil of Keynes, Ramsey sympathetically criticized Keynes' work on mathematical probability. Keynes, as the editor of *The Economic Journal*, promoted Ramsey's two economics essays, on the optimal rate of taxation (1927) and savings (1928). Modern economics graduate students (at least at MIT) are familiar with variants of these "Ramsey models".

The Ramsey (1928) essay on the optimal rate of saving expressed the Cambridge concern with social welfare. Moreso, as Keynes commented,

> (i)t is a remarkable example of how the young

> can take up the story at the point to which the previous generation brought it a little out of breath, and then proceed forward ...

(Keynes, 1933, pp.336-37). The essay assumed the utility maximizing paradigm introduced by Jevons (1871) and Walras (1900). Utility as the measure of material welfare was cardinal and additive. Capital was a produced good and the labor force, which was assumed to be employed fully, was exogenously given. A net increase in capital goods (or saving) resulted from abstaining from consumption out of current income. In return for abstaining, accumulation enhanced the production of future consumer goods. Only consumer goods yielded utility, with the real value of each good measured by the marginal utility of consumption. Given this framework, the question naturally arose about that division of each period's income between saving and consumption that maximized utility over intervals of time. Ramsey, in contrast to his predecessors, thought about this question mathematically.

The static, marginal utility paradigm assumed a convex production function. There was no notion of technical progress, so that increases of capital relative to a fixed factor at the limit yielded a zero net marginal product, with the result that the economy approached a stationary state. The utility function was also convex, so, assuming static preferences and a fixed population, at the limit marginal utility approached zero, the point of saturation. Ramsey, having adopted the principle of diminishing returns, modelled the time-path of capital that maximized total utility.

The problem of maximization in economics originated as an analogy to minimization in Newtonian mechanics, which was solved by means of differential calculus. The first mathematical economist, Cournot, was a mathematician by training. While Cournot's economics (1838) presented the rudiments of a static equilibrium theory, it lacked a concept of utility. Later, Jevons and Walras, who were utility theorists, were stymmied in their attempt to elaborate a mathematical economics because they knew little calculus. The contributions of economists with formal training in advanced calculus like Edgeworth, Marshall, Wicksteed or Ramsey greatly advanced the field.

Just as Jevons and Walras set up the problem of a static equilibrium as an analogy to Newtonian statics, so Ramsey proposed a mechanical theory of economic dynamics. In particular, his solution to the problem of the optimal path of savings followed marginalist principles modelled after Newton's third law -- that to every action there is an equal and opposite reaction.

It was Keynes who explained to Ramsey the rule of optimal saving arrived at from marginalist economic reasoning. After all, Keynes was by origin a Marshallian familiar with neoclassical principles of saving. Meanwhile, Ramsey made savings depend on the propensity to save and income, rather than on the rate of interest, which followed Keynes' dictates. The main difference between them was that Ramsey assumed an equilibrium, with savings equal to intended investment.

Ramsey posed the infinite horizon problem of the maximization of social utility subject to contraints as follows: What rate of savings, $\dot{k}$, would minimize the difference (J) between the level of at bliss (B) and the sum of each instant's utility (U(C)), given the disutility of labor (V(L))? -- that is, in mathematical terms,

$$(10.)\qquad {}_{\min}J = \int_0^\infty \frac{[B - U(C) + V(L)]}{\dot{k}} dK, \qquad U'(C)>0,\ U''(C)<0.$$

There was no discount factor, or rate of time preference since the social planner, who had perfect foresight, equally weighted the utility that accrues in successive periods. Bliss B defined the upper bound of the integral, so that the integral converged. The level of utility was constrained by the scarcity of resources, as indicated by the accounting identity,

$$(10b.)\qquad \dot{k} \equiv F(K,L) - C, \qquad F'>0,\ F''<0,$$

where F was a constant returns to scale production function and the initial stock of capital is positive, $K(0)>0$.

Ramsey solved the minimization problem by setting the derivative of the integrand with respect to the independent variable $\dot{k}$ equal to zero. This yielded what has become known as the Keynes-Ramsey rule:

$$(11a.).)\qquad U'(C)\ \dot{k} = -(B + U(C) - V(L)).$$

This meant that, along the optimal path, as the economy neared bliss, the marginal utility that resulted indirectly from investment in any period (the left-hand side) was just sufficient to make up for the difference between bliss and the instantaneous level of utility (the right-hand side). This implied the terminal condition of zero marginal utility and/or a zero marginal product of capital. In this case, bliss was a stationary state in which utility was constant.

Ramsey also solved the minimization problem (equation 10a) by means of the calculus of variations. This yielded the Euler differential equation

$$(12.)\quad F'(K)\, U'(C) + \mathring{U}'(C) = 0,$$

which required that the proportional fall in the marginal utility of consumption equall the marginal utility of the marginal product of capital. This equation is the form taught today as the "Keynes-Ramsey" rule [Solow [1980]; Blanchard and Fisher (1989)].

In Ramsey's time, economics was largely literary in style. Many readers of *The Economic Journal* would not have appreciated this mathematical essay on economic optimization. As Keynes (1930) commented in his eulogy of Ramsey,

> 'A Mathematical Theory of Saving' ... is ...
> one of the most remarkable contributions
> to mathematical economics ever made, both
> in respect of the intrinsic importance
> and difficulty of its subject ... The article
> is terribly difficult reading for an economist

(p.335-36).

Subsequently Keynes repudiated the formal framework of utility maximization in *The General Theory*. Yet economics, in the decades following Ramsey's essay, became highly mathematical in content. In this context, Ramsey's mathematical law was rediscovered. Samuelson, praising the precision of mathematics, announced that the economy that followed the optimal path would

> Accumulate! Accumulate! Accumulate!
> But not faster than Ramsey's Law ...
> (n)o loose statement can do justice to
> the Theorem, which says what it says,
> not more and not less

(1965, p.494).

In the new edition of *Foundations*, Samuelson (1983) even stated the Ramsey problem of maximizing utility over infinite time in three ways.[16] He stated the problem in terms of the integral,

(13a.) $$\max_{K(t)} J = \int_0^\infty U[F(K)-\dot{K}]\,dt,$$

without the device of bliss. He stated the Ramsey problem by means of the "energy integral,"

(13b.) $$H = \max U[F(K) - \dot{K}] + U'[F(K) - \dot{K}]\dot{K} = \overline{c}, \qquad \overline{c}=0.$$

This Hamiltonian notation meant that the optimal program makes instantaneous utility equal the marginal utility yielded by investment. He also specified the Ramsey problem by means of the "Hamilton-Pontryagin" function,

(13c.) $$H(p,K,\rho t) = \max_{\dot{K}} e^{-\rho t}U(\dot{K},K) + \Sigma_1^n p_j \dot{K}_j, \qquad p=U'Ce^{-\rho t}.$$

p was the shadow price of investment expressed in terms of marginal utility and ρ was the social rate of discount. This function maximized the present value of utility in the first instant and the sum of utilities yielded by investment.

These Hamiltonian functions implied an analogy between the conservation of energy and utility, as shown in Table 2.

Table 2.

Classical Mechanics: $H(x,p;t) = -L(x,\dot{x};t) + \Sigma p\dot{x} = \bar{c}$

Samuelson (1983): $H(K,U'(C);t) = U(K, \dot{k};t) + \Sigma U'(C)\dot{k} = \bar{c}$.

This formal mapping gave an analogy between momentum, p, and marginal utility, U', and the position of a particle, x, and the capital stock, K. It meant that utility in economics like energy in mechanics is conserved. But in physics, the Hamiltonian played an important theoretical role. In economics, the conservation of utility has had no theoretical significance.

The use of analogies was common in the physical sciences. Analogies established a formal comparison between two systems, with the purpose of suggesting theoretical claims to be tested. Testing typically revealed positive, but also negative analogies, which inspired further research. In this process, analogies were truly theory constitutive. In contrast, in economics, analogies existed as formal mappings that merely legitimized existing economic constructs.

The first edition of *Foundations* (1947), Samuelson left out the problem of the optimal path of saving. The first edition concerned the existence and the stability of the static, competitive general equilibrium system. The second edition (1983) treated the problem of dynamic optimization in terms of the formalisms of the physics of energy. What happened between the publication of the two editions to explain this change in perspective?

C. Samuelson and Solow: Precursors of the New Growth Theory

Economists took a long time before focusing on the problem of dynamic optimization. They began to come to grips with the Walrasian problem of maximization in a static general equilibrium system only in the 1930s. Cassel's (1932) Walrasian textbook, the influx of former physicists into economics, the promotion of mathematical economics at the Cowles Commission, and Hicks's (1939) response to the Keynes' general model together triggered a concerted research effort. Over the next two decades, neoWalrasian economists demonstrated the conditions for the existence and the stability of the static general equilibrium model.[17]

Samuelson had the training and the interests to play a leading role in the Walrasian revival. As he said, "I was lucky to enter economics in 1932." In the 1930s, Samuelson formed a "master"-student relationship with E. B. Wilson, the author of a textbook (1912) mathematical physics and became a "disciple" of J. W. Gibbs. From Wilson, Samuelson gained that faith in the unity of science that marked the Walrasian project from the start. As the enlarged edition of Samuelson's *Foundations* (1983), in a section entitled "Newtonian Paradise Regained", narrated,

> the Weak Axiom of Revealed Preference ... follows from the basic logic of maximizing. One of the most joyful moments of my life was when I was led by listening to E. B. Wilson's exposition of Gibbsian thermodynamics to infer an eternal truth that was independent of its physics or economics exemplification. (A student who studied only one science would be less likely to recognize what belongs to *logic* rather than to the nature of *things*)

(pp.xviii-xix).

From this perspective, Samuelson originally intended *Foundations* to establish the conditions of economic equilibrium as those of the extremum problem that arose in classical dynamics and Walrasian economics. By defining the problem of production, cost and demand as one of maxi(mini)mization, Samuelson intended to bring all economic theory under the rubric of a few basic principles, just as thermodynamics revolved around just a few laws. The focus of the book concerned the formal dependence between comparative statics and (short run) dynamics, known as the Correspondence Principle. In particular, Samuelson purported that the restrictions imposed on the stability conditions given a change in the parameters of the system would have implications for the properties of equilibria which were empirically refutable. Hence, economics like thermodynamics would be a mathematical, deductive system that was operationally meaningful.

Foundations gave one of the earlier applications of the Lagrange multiplier technique to the case of economic maximization subject to constraints. It established conventions regarding a host of issues, including the properties of stable systems and the formal underpinnings of the consumer demand curve. In retrospect, one might say that the economics graduate student who, according to the minimal requirements, mastered the first four chapters on statics and comparative statics in Chiang's six-chapter mathematical economics textbook (1984) could understand most of the first nine of the eleven chapters of *Foundations*.

Yet, Samuelson (1983) saw that

> by the time *Foundations* celebrated its official twentieth birthday, its pages of Newtonian calculus were old hat.

(p.xviii). Samuelson had in mind that *Foundations* (1947), given its focus on statics, contained little by way of

> "the fashionable Hamiltonian formalisms that are often used in the physics and mathematical literature to describe variational problems"

(Samuelson and Solow, 1956, pp.554-555).

Only the last two chapters of *Foundations* (1947) addressed the problem of the stability of a dynamic system. Samuelson distinguished between two types of dynamic systems, one "historical", the other "causal", or "nonhistorical". The historical system was nonstationary, nonconservative, and time dependent, with the result that motion was nonreversible. A nonhistorical system was stationary and conservative. This system was a "complete causal system" in the sense that the knowledge of the initial conditions of a system and the time that elapsed since those conditions was sufficient to determine the position of any variable in the system.[18] The same initial conditions later in time would generate the same evolution of the system, except at continually later time period. Economists should think of this nonhistorical system as being always in equilibrium, just as engineers using variational calculus,

> think of a cannon ball as being in equilibrium, not only after it has fallen to the ground at rest, but also at every point in its flight, when it is on its mean trajectory as well as in its precession around this path

(pp.331-32). In the 1950s, Samuelson and Solow pursued this analogy in a model of the trajectory of capital over the infinite horizon by applying the formalisms of energy physics.[19]

Samuelson wrote a paper on dynamic optimization with heterogeneous capital (1956) and aggregate capital (1960). Although Solow's name was not attached to the latter paper, he was involved in the whole project of applying variational calculus to model capital. Solow, who had recently completed his Ph.D.

dissertation at Harvard and joined Samuelson at the MIT, wrote an unpublished draft of the 1960 paper. This paper originally was intended as a chapter in the Dorfman, Samuelson and Solow textbook, *Linear Programming* (1958). Both this book and the 1960 paper was financed by the RAND Corporation, a special group inside the Douglas Aircraft Corporation during World War II and a corporation since 1947, which gave most of its contracts to air force operations analysis.[20] In the Cold War period, a sound program of economics also was viewed as an important part of the national defense. Linear programming, which the Air Force developed during the war to analyze its interrelated activities, would extend the scope of economic applications.

The economic research on optimal capital accumulation was linked directly to the research interests of the Air Force, though the Office of Naval Research sponsored some of the economics projects as well. In the early 1950s, the Air Force ran a research program on the control of the trajectory of air-weapons. The major output of this program was the RAND publication *Dynamic Programming* by Bellman (1954). The book modified variational calculus to model

> an optimal policy [that] has the property that whatever the <u>initial state</u> and <u>initial decision</u> are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision

[p.83].[21] This property of the optimal control of an air-missile was identical to that of the nonhistorical economic system defined by Samuelson in *Foundations* (1947) and essential to the dynamic optimal control model that Samuelson with Solow later initiated.

Bellman (1954) began his analysis of the optimal policy for missiles by presenting the classical brachistochrone problem.[22] In classical calculus, or the calculus of Lagrange, as he explained, the problem was to determine the entire sequence of moves constituting the quickest descent. In contrast, Bellman took a "new approach". As he stated,

> (a)n advantage of this new approach lies in the fact that very often in the determination of optimal policies for multistage processes, the determination of the next move in terms of the current state of the process is in many ways a simpler, more natural and even more important piece of information

for planning purposes (p.248). The new formalism that Bellman developed, known as dynamic programming, differed only in detail from the classical Hamiltonian method.[23] Its early applications programmed the multi-stage transition of a missile along a path so as to minimize a function of the final state variables (such as the maximum payload of the Apollo spaceship).[24] Solow and Samuelson (1956) followed Bellman's recursive procedure in modelling the path of capital.

Samuelson and Solow relied on a number of other sources on the calculus of variations. Samuelson at least was familiar with the second edition of *Dynamical Systems* (1927) by Birkhoff, the leading mathematician at Harvard. They knew parts of Carathéorydory's calculus book (1935), either directly or through citations in the Bliss *Lectures* (1946).[25] They drew heavily from Bliss, who supervised a series of Ph.D. dissertations (1930-1942) in Mathematics at the University of Chicago on problems in the calculus of variations.

It might seem surprising that Samuelson and Solow, who were amongst the architects of the neoclassical synthesis, should develop a line of research that assumed the economy to be continuously in equilibrium. But scientists typically broach questions that interest them within the horizons of a broad research program, without initially being concerned about consistency. Thus Samuelson explained that the details of the economic system that a researcher defined depended on the "purpose at hand".[26] Both economists thought that if the real economy worked ideally, it would exhibit the properties of a general equilibrium. Moreover, the question that Samuelson and Solow posed about the optimal path of the economy followed an easily recognizable line of thought within standard economics. Samuelson's (1948) principles textbook, the prototype for the next three decades, introduced the instantaneous production-efficiency frontier and explained to students that the economy that maximized would locate at this frontier. Most of the principles course went on indicating the conditions that a static economy had to satisfy to reach that frontier. Dorfman, Samuelson and Solow (1958) then introduced the <u>intertemporal production-efficiency frontier</u>, using linear programming to show the condition that the efficient economy satisfied within an interval of time. By repeating this calculation in respect to successive intervals, in principle, one could arrive at the entire path of the efficient economy.[27] Samuelson and Solow (1956) this problem, posed earlier by Ramsey, by means of the Hamiltonian calculus.

Ramsey (1928), *Foundations* 1947, RAND in the 1950s ... Thereisone more strand in this story, the issue of whether capital in the aggregate could be thought of as only one good. The technique of linear programming maximized a function of a number of variables subject to constraints in the form of inequalities. In devising this technique, Dorfman, Samuelson and Solow (1958) intended to improve upon the disaggregated Leontief model, with its fixed coefficients of production. In an optimization framework, they generalized the production function and the cost function to the case of many consumer and capital goods. Similarly, Samuelson and Solow in their (1956) paper extended the Ramsey model to the case of numerous consumer and capital goods. They then demonstrated that the necessary condition for optimality over time was identical in the cases of one capital good and many capital goods. This result appeared to confirm Samuelson's (1947) assertion that "logically" mathematics would not distinguish between the cases of (1) the representative good and agency and (2.) numerous goods and agencies.[28]

Whether or not an economy with heterogeneous goods optimized over time remained a theoretical issue in the 1950s.[29] Samuelson and Solow (1956) knew about the criticisms of the aggregate model of capital made, more forcefully than Wicksell and Hayek, by economists at Cambridge, England.[30] In particular, J. Robinson (1953-54) argued vituperatively that it was impossible meaningfully to aggregate heterogeneous capital goods, because the value of capital varied with the rate of interest. It was the Samuelson and Solow position in 1956 and throughout the capital controversy that though (i.) capital goods were physically heterogeneous; nevertheless, (ii.) the model of an efficient dynamic economy required perfect knowledge of future rates of interest and valuations of capital; so that (iii.) working with the Ramsey model of abstract, aggregate capital, calculated in value terms, produced basically the same solution as the heterogenous capital model.[31]

The Samuelson and Solow essay on a heterogeneous dynamic capital model was one of the earlier responses of the economists of Cambridge, U.S. during the capital controversy. In addition, the essay was intended to revive the Ramsey utility-maximizing model. Yet, this paper has remained amongst the least known publications of either author. The problem was mainly a matter of mathematics. The authors failed to explain the meaning of the Hamiltonian formalism, with which few economists at the time were familiar. In addition, the notation was often baroque and incomprehensible in economic terms.

The essay began by reviewing Ramsey's original (1928) problem, which was restated in the form of the maximization problem,

$$(14.)\qquad {}_{\max}J = \int_{b}^{a^*} \frac{U\{F(K) - \dot{K}\}}{\dot{K}}\, dK, \qquad \begin{array}{l} U'>0,\ U''<0 \\ F'>0,\ F''<0, \end{array}$$

where the utility and production functions both exhibited diminishing returns given an increase of the capital stock. The end point a* signified the point of bliss, where the marginal product of capital was zero and investment ceased, the case of the stationary state. Substituting time t instead of $\dot{K}$ as the independent variable in equation 14 and differentiating in respect to time yielded the Keynes-Ramsey rule.[32]

The authors interpreted instantaneous utility U to denote independently additive, or cardinal utility. This left the well known difficulties of measurement summarized by Samuelson (1947) in his development of the doctrine of revealed preference.[33] Because of these difficulties, Samuelson and Solow would have preferred approaching the problem of intertemporal efficiency without a complete model that specified preferences. Samuelson's (1960) paper on the efficient paths of capital accumulation and the Solow growth models avoided this difficulty by ignoring the

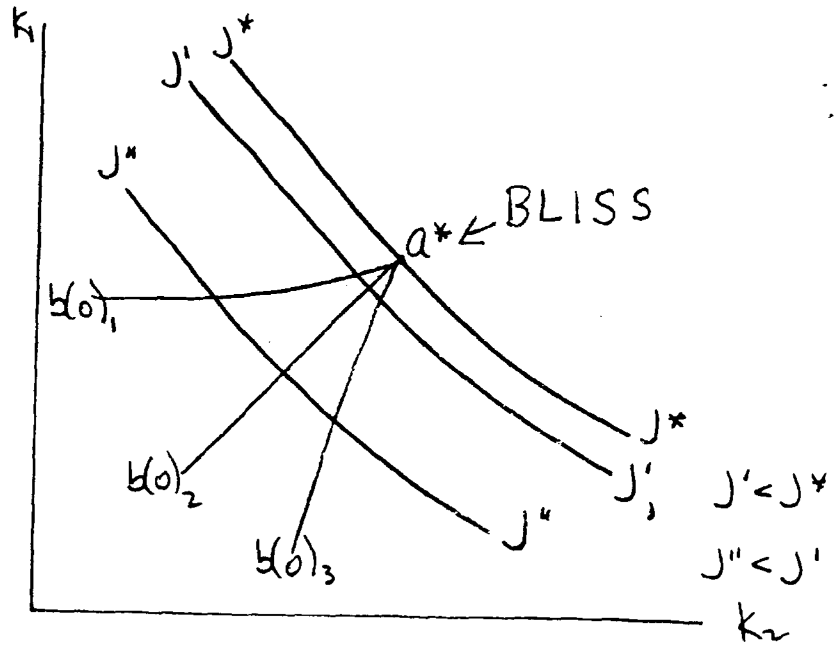


Figure 1.

(Samuelson and Solow, 1956, Figure 3)

utility integral.[34] The 1956 exercise, as a precursor of the later neoWalrasian dynamic utility maximizing models, was not typical of the work of either author.

The authors restated the utility maximizing problem in terms of heterogeneous capital goods (K_1, K_2). Their problem was to show that all paths of capital that satisfied the Euler condition would approach the unique point, bliss (a_i*) because any path that strayed in another direction could be bettered. They illustrated this problem in Figure 1, where the J curves described the composition of capital (K_1/K_2) that maximized utility at each state. (Samuelson (1960) found the curve J* to be the "curve of 'steepest descent' or the brachistochrone.[35] The growth of the value of capital evaluated at constant market-clearing prices was maximized along this curve.) Given the alternative initial conditions ($b(0)_{1,2,3}$), each point along the paths from b to a* gave that composition of capital required for the subsequent optimal provision of consumer goods. At bliss (a*), utility was maximized and no change in the composition of capital could increase the marginal product of capital in terms of consumption.

The solution of the optimal path was a difficult problem. The authors preferred to use "the fashionable Hamiltonian formalisms" to solve it, though these offered no computational advantage.[36] They mapped out the economic formalisms onto the Hamiltonian calculus. They defined the Lagrange function L as a constant,

$$L \equiv T - V,$$

where T was kinetic energy and V potential energy.[37] We know from nineteenth century mechanics that kinetic energy was thought of as the integral of force, or

$$T = \tfrac{1}{2} m \Sigma \dot{x}^2_i = \tfrac{1}{2} p \dot{x},$$

where m stood for the mass of identical particles, $\dot{x}_i$ for the velocity of the particles and p for their momentum.[38] Kinetic energy was defined as

$$T = \tfrac{1}{2} \Sigma\Sigma \alpha_{jk}(K_i)\dot{k}_k \dot{k}_j .$$

The authors did not define the coefficient a_{jk}, which presumably gave the quantity of input j to produce one unit of output k. What precisely this had to do with the concept of a physical mass remained mysterious. $\dot{k}_k \dot{k}_j$, which stood for the product of investment in two types of capital good, was, in mathematical terms the product of two derivatives. The analogy between this and the square of the velocity of a particle also was mysterious. We also know that classical mechanics defined potential energy in terms of position, or the coordinates

$$V = V(x_i).$$

Samuelson and Solow too defined potential energy in terms of the coordinates

$$V = V(K_i).$$

The authors then argued that the economic system that they constructed was a conservative one if the variable, K_1 was not independent, but was determined by n-1 K variables.[39] However, the existence of a conservative system depended on whether the Hamiltonian T + V was a constant, not whether there was a general equilibrium system in which one variable was redundant.

Samuelson and Solow next defined "momenta", given the marginal rate of substitution between the two capitals, K_j, K1,

(15.) $p_j = \partial L/\partial K'_j$, $K_j' \equiv dK_j/dK_1$.

However, in Hamiltonian mechanics, momenta was due to the fact that the Lagrange integrand differed from place to place as you varied the velocity of a particle.[40] The Samuelson-Solow notion of momenta (p_j) did not involve velocity and was not analogous to the mechanical concept of momenta.

The Lagrange integral L, stated in terms terms of the stock of heterogenous capitals and the marginal rate of substitution between these capitals, was an incredibly cumbersome expression with no apparent economic meaning. The Hamiltonian function was

(16.) $H(K_i,p_j) = - L(K_i,K_j') + \Sigma_2^n p_j K_j'$. Since the Hamiltonian took on a constant value, the economic system was "truly time-free" (Samuelson and Solow, 1956, p.556).

Hamilton's canonical equations gave Samuelson and Solow 2(n-1) first order differential equations for 2(n-1) independent variables (K_j, p_j) that solved the multi-stage decision rules that told the system how to invest and consume. One could work via a sequence of solutions backwards from bliss or forwards from take-off, since the variational system was reversible. The amount of information required at each stage-- the quantity of each type of

capital and the marginal rate of substitution between capital --
was, in the restrained words of the authors, a "tall order".[41]
Assuming that the computing capacity existed, the initial solution might be carried out by a central planning agency or by,

> a research department, not because its computation saves labor, but because its information can be conveniently turned over to the line officials who currently make decisions at each given state of the system (b(1),...,b(n-1),a*) ...

(pp.558-559). In retrospect, historians of economics may have been prone to classify this paper as a piece of Cold War science fiction, written for economists. In fact, as Burmeister and Dobell [1971] concluded in their survey of mathematical growth theory, the paper turned out to be

> a remarkable anticipation of the maximization principle and of a literature to burgeon a decade later [.] Samuelson and Solow suggest many of the foregoing ideas in their analysis of the Ramsey problem with heterogeneous capital goods

[p.404].

III. The Solow Growth Model

As the Ramsey models that he introduced burgeoned in the 1980s, Solow disavowed the models as "far-fetched", stating that

> the dynasty is supposed to solve an infinite-time utility-maximization problem ... The next step is harder to swallow in conjunction with the first. For this consumer every firm is just a transparent ... device for carrying out intertemporal optimization subject only to technological constraints and initial endowments. Thus any kind of market failure is ruled out from the beginning, by assumption

(1988, p.310). Solow, as years passed, adopted a different philosophy. Any mathematical economic theory was a simplification, but he intended his growth theory itself to incorporate "the first few doses of realism".[42] The motivation behind Solow's philosophy was two-fold. He clearly intended his growth economics to feed directly into macroeconomic policy-making. Connected with this, he believed that the bare tenets of orthodox theory were true and to be defended. Thus, his work became caught up in the Cambridge controversy, which boiled down to a dispute over the realism of the marginal productivity theory of income distribution. Given the mathematical problems of incorporating realistic features into a general equilibrium model, Solow's models characteristically were incomplete, appearing as they did in aggregate form and without any preference functions.

Neoclassical growth theory arose in the latter 1950s as a challenge to the dominant paradigm proposed by Harrod, who was a member of the economics department at Oxford and Britain's most prominent Keynesian in the early postwar period. Harrod's model purported that the economy even in the long run failed to approach a full employment equilibrium path.

Harrod, who during a term at Cambridge in the 1920s came under Keynes' tutelage, generalized the investment multiplier presented in the context of the short run dynamics of *The General Theory* to the case of long run growth in "Towards a Dynamic Economics".[43] The static investment multiplier was defined as

(16.) $Y/I = 1/s$,

where Y stood for output, I investment and s the propensity to save out of income. Dividing the left hand side of this multiplier by ΔY gave the economic growth rate G in terms of the propensity to save and a <u>variable</u> capital-output ratio, C, that is,

(17.) $G = s/C, \qquad G \equiv \Delta Y/Y, \; C \equiv \Delta K/\Delta Y \equiv I/\Delta Y, \quad s \equiv S/Y$,

or Harrod's dynamic equation. There were three types of growth rate G. The growth rate of a fully employed economy, or the natural rate of growth G_n, was determined by the growth of the labor force and the pace of technological improvement. The growth rate of an economy in profit-maximizing equilibrium with unemployment, the warranted rate of growth G_w, was determined by entrepreneurs' propensity to save s and the desired capital-output ratio *C*, where capital included inventories. The actual rate of growth G_a deviated from G_w when investment plans were disappointed and the economy was in disequilibrium. Harrod, who adopted the arguments of *The General Theory*, detailed the "centrifugal forces" that, in the presence of economic uncertainties, continually pushed capital investment off its long run equilibrium path, so that the warranted rate of growth showed no tendency to equal the natural rate.[44] Thus the variability in the capital-output ratio was insufficient to secure a stable equilibrium with full employment. The source of instability did not lie in the production technology, but in the economic circumstances that determined the relation between the rate of interest and the expected rate of profit on capital.

In one of the more remarkable cases of simultaneous discovery, Swan (1956) and Solow (1956) a few months before, assuming the standard neoclassical model, showed by means of a phase diagram (the sort used in mechanics) that forces caused the system to approach a stable long run equilibrium. In contrast to Harrod's model, in this model the warranted rate of growth appeared to approach the natural rate.

Solow defined the static, aggregate production function

(18.) $Y=F(K,L_0e^{nt})$, $F'>0$, $F''0$,

where F was Cobb Douglas and n, the constant, exogenous rate of growth of the labor force. The assumptions of flexible returns to labor, profit maximization and perfect foresight of future rentals and interest rates assured that the labor force and the capital stock were employed fully.

The assumption that firms maximized profits in the absence of risk and uncertainty removed the conditions under which unstable growth arose in the Keynesian framework. The task of proving that the economy was stable in the long run became a mere problem of modelling the technique of production. As Solow stated,

> this fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of *fixed proportions*. There is no possibility of substituting labor for capital in production.

(p.65).

Solow differentiated equation 18 with respect to time and divided by L_0e^{rt}, which yielded the equation

(19.) $\dot{k} = \bar{s}f(k,1) - nk.$

k stood for the capital-labor ratio and f for the total product as varying amounts of capital were employed with one unit of labor. sf meant savings, determined by the constant propensity to save outof income. nk gave the level of investment at any capital-labor ratio given the constant rate of growth of the labor force. The equation said that when savings equalled investment, the capital stock expanded at the rate of growth of the labor force. In other words, growth was balanced.

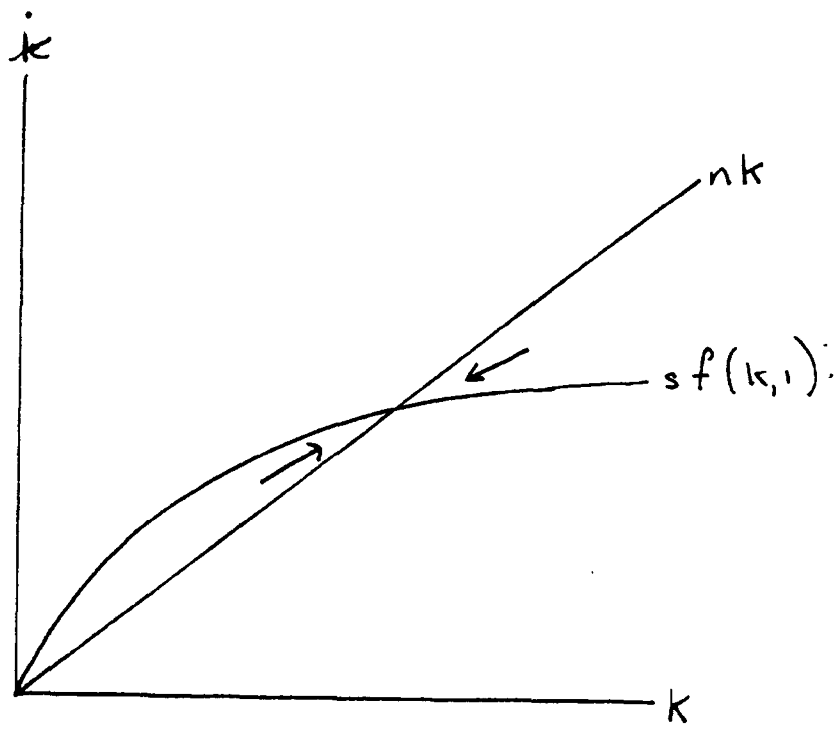


Figure 2.
(Solow's 1956 Figure 1)

My figure 2 (Solow's figure 1) shows the capital-labor ratio adjusting to the equilibrium ratio that was consistent with the equality of (i.) savings out of income and (ii.) desired investment. The savings curve ($\bar{s}f$) sloped downwards because the production function exhibited diminishing returns to capital. The equilibrium point was a stable one because the investment curve (nk) crossed the savings curve from below.

Solow's figure exemplified the <u>timeless</u> phase portraits that economists borrowed from rational mechanics.[45] The approach to the equilibrium point would be asymptotic in <u>infinity</u>, with the speed of travel toward it directly related to the distance remaining to be travelled. This would be apparent were figure 2 transformed in terms of a first-order differential equation in the variable $\dot{k}$. Phase portraits became an obligatory in any dynamics around this time.

In Solow's figure, the stable capital-labor ratio (k*) was consistent with that ratio of the rate of profit to the real wage that secured full employment. When the capital-labor ratio was less than the equilibrium level, the ratio of the rate of profit to the real wage was relatively high and falling, and conversely, when the capital-labor ratio exceeded the equilibrium level.

The neoclassical geometry parodied the conventional Keynesian short run analysis of adjustment to equilibrium in the presence of unemployment. In the context of the 45° line diagram, the Keynesian multiplier principle defined economic equilibrium in terms of the savings-investment relationship. When the economy was in disequilibrium, a higher ratio of the under-utilized capital stock to workers occurred when planned investment exceeded savings. A relatively low ratio of under-utilized capital to workers occurred in the converse case. But the Keynesian analysis made the given ratio of the capital stock to labor and the equality of planned investment and savings consistent with any level of employment of labor and capital. Solow simply mapped the Keynesian short run disequilibrium model of the savings-investment relationship and the variations of the capital-worker ratio onto his graph showing the adjustment of the capital-labor ratio to the neoclassical long run.

It is obvious that Figure 2 (Solow's Figure 1) contained an error. Treating the diagram like a phase portrait, Solow plotted the position of the variable k against its motion, $\dot{k}$. However, in economic terms, the capital-labor ratio k was constant in equilibrium, i.e., $\dot{k}=0$. The vertical axis should not have been labelled $\dot{k}$, but nk ($nk \equiv I$). The same error appeared on four other figures in the 1956 essay. So much for Solow's youth and the refereeing of journal articles in those days. The rhetorical power of the geometrical style of argument was so great that one major mathematical economics textbook copied this figure *verbatim*.[46]

Solow recognized that the geometrical stability proof rested on the specification of a diminishing returns production function. As he stated

> (o)f course the strong stability shown in Figure 1 [my Figure 2] is not inevitable. The steady adjustment of capital and output to a state of balanced growth comes about because of the way I have drawn the productivity curve f(k,1). Many other configurations are *a priori* possible

(p.73). In the case of a production function with increasing returns to scale, the savings curve could lie to the left of the investment curve, in the range of practical capital-labor ratios. The requirement that the geometrical model produce a long run stable solution ruled out a growth theory with increasing returns to scale.

The constant returns to scale production function could not account for the historical rate of growth of productivity. With the Cobb Douglas function, increases in productivity depended solely on increases in the capital-labor ratio, which historically were too small to explain productivity growth. So Solow, like Tinbergen (1942), inserted a multiplicative, exponential term into the production function,

(20.) $Y = (L_o e^{nt})^{\alpha} K^{1-\alpha} e^{rt}, \qquad 0<\alpha<1.$

which meant that the level of technology varied with time given the rate of scientific progress.

This specification of the source of productivity growth had two convenient features. It left the corpus of microeconomic theory intact and provided a tool with which to organize time series data as a basis for short run demand management.

In a perfectly competitive economy, the output elasticities α, $1-\alpha$ stood for the income shares of labor and capital. Technical progress, which made these factors more productive, received no reward. The neoclassical model therefore treated technical progress as a nonexcludable, public good, as if research and development and patents on knowledge did not exist at all. Modelling technical progress this way left the marginal productivity theory that accounted for pricing and income distribution entirely intact. Since technical progresss did not arise from a process of learning, the model of the growing economy was path and time independent.

From within the neoclassical framework, the introduction of the trend of technology created a modelling difficulty. Since technical progress was made a multiplicative factor, algebraically it was possible to make progress augment capital or labor or be neutral. In the original notation, Solow and Swan made technical progress augment capital, which resulted in capital growing faster than labor, or unbalanced growth.

Solow's discussion of this technical problem was replete with algebraic errors, which he later corrected.[47] Again, for most readers, the mathematical details went unnoticed. The credentials of the author, the style of writing and the apparent reasonableness of the intended meaning together permitted his major point to come across. The precise mathematics served as ornamentation.

To preserve balanced growth given technical progress, it was necessary to make progress labor augmenting. New knowledge increased the efficiency of each unit of natural labor by a factor of e^{rt} and fixed the marginal product of capital at any given capital-output ratio. When balanced growth was Cobb-Douglas, balanced growth was consistent with neutral or labor augmenting technical progress.[48]

The whole issue of specifying the constant returns to scale production function with technical change concerned mathematical notation rather than the real effect of technical progress. So Solow (1970) remarked that,

> (i)t should be realized that this reduction of technological change to the efficiency-unit content ... of labor is a metaphor. It need not refer to any change in the intrinsic quality of labor itself...What matters is this special property that there should be a way of calculating efficiency units of labor *dependent on the passage of time but not on the stock of capital,* so that the input-output curve doesn't change at all in that system of measurement

(p.35). Thus, to maintain a model of balanced growth, which ensured stable growth -- the purpose of the neoclassical attack on Harrod -- Solow and Swan adopted an "anti-accumulation, pro-technology" line of argument (Swan, 1956, p.338). Changes in the propensity to save (or invest in capital) increased the speed at which the economy approached the steady state. In the steady state, the propensity to save effected the level of output, but not its rate of growth.

Solow, Samuelson and other neoclassical economists maintained that the economy only tended to full employment. Given exogenous disturbances to aggregate demand, rigidities in money wages and interest rates prevented an immediate return to full employment equilibrium. In the short run, the economy moved in a cycle of boom and slump. Economic statisticians since the 1930s extracted a linear trend from the time series data, which left a data scatter with a cyclical shape. Solow (1956) and others lent legitimacy to this statistical technique by defining the linear trend in a theoretical context.

In 1961, the Kennedy Council of Economic Advisers adopted the neoclassical model as a basis for demand management and policies towards growth. Solow, who was a member of the CEA staff, helped write the report of the Council.[49] The trend of output and augmented labor was estimated to be 3 per cent a year. The full employment rate of unemployment was put conventionally at 4 per cent, on the assumption either than this rate was consistent with zero inflation or full capacity utilization. As shown in figure 3, the full employment benchmark and the long term rate of growth

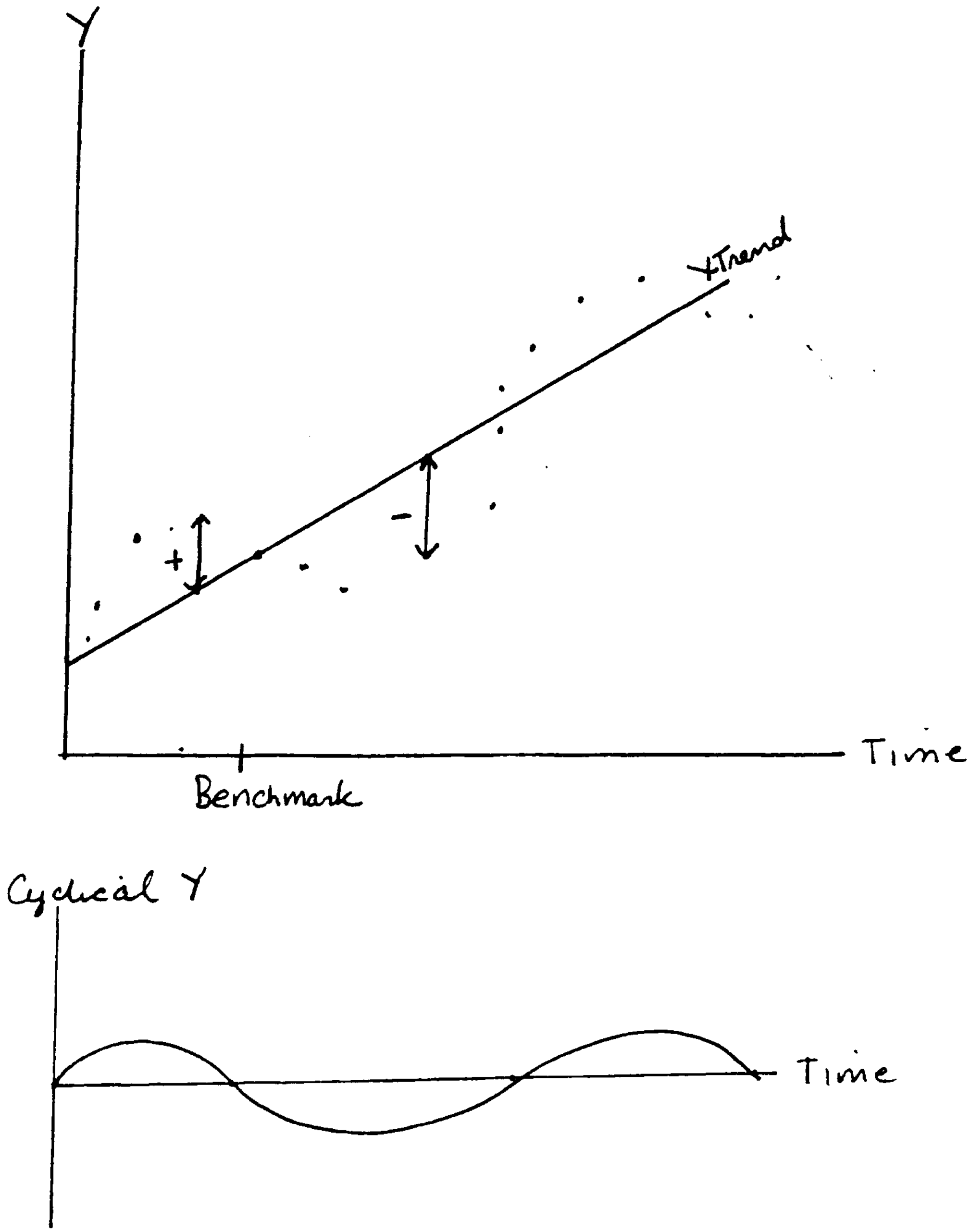


Figure 3.

Exogenous Trend and Time Series Organization.

together gave the trend line of output. The positive and the negative deviations from this trend exposed a short run, roughly stationary, cyclical motion.

The CEA's estimate of the rate of technical progress was based mainly on Solow's research. Solow (1957) specified the Cobb Douglas production function with exogenous technical progress. This function made changes in labor productivity a function of changes in the capital-labor ratio (ignoring the issue of balanced growth and augmented labor) and the level of technology, given the output elasticities of labor and capital. How could one separate out the shift in the production function due to technical progress and the movements along the production function due to changes in the capital-labor ratio? Assuming perfect competition, so that the output elasticities were identified with factor income shares, Solow put the Cobb Douglas production function in per capita growth terms and calculated the rate of technical progress as the difference between the growth of labor productivity less the product of capital's share of income and the growth of the capital-labor ratio. The estimates were done on the time series 1909-1949, with the value of capital referring only to capital in use. Given the questionable assumption that changes in productivity due to cyclical variations averaged out, Solow concluded that knowledge increased by 2 per cent a year and technical progress accounted for 87 per cent of labor productivity growth, a result that fell close to other estimates of the time.

Solow, like other mainstream economists at the time, was unsatisfied with the artifice of exogenous technical progress. Besieged by the criticisms of Joan Robinson and Kaldor about the production function, Solow was even more eager to improve the growth model, with the proviso that he maintain the marginal productivity theory of income distribution. Writing in 1959 on fiscal policies to encourage investment, he complained that the model made technical change "float down from the outside" as if "peculiarly disembodied". The time shifts of the production function that he estimated in 1957 were "a confession of ignorance". The "practical question" concerned the effect on of increases in the growth of capital on productivity growth.[50]

Solow emphasized that the 1957 estimates rested on the measurement of capital, a task replete with pitfalls (Solow, 1957). He agreed with economists who said that the notion of exogeneous technical progress was " measure of our ignorance".[51] Denison (1962) reduced the residual by making upward adjustments in the measurement of inputs in the service sector. Jorgensen and Griliches (1967), who also corrected the value of capital services, virtually eliminated the residual.

Solow (1959) sidestepped the problem of explaining the residual by embodying exogenous technical progress in capital. He specified a Cobb-Douglas production function where technical progress augmented capital, defined as the integral of past investments. The long run rate of growth of productivity depended on the embodiment of knowledge in new capital and thus on the rate of investment. There was an exogenous upper bound to the rate of technical progress, which determined the shifts of the production function. Investment, however, might be too low because of market failure. In this case, government intervention was required to increase the rate of technical progress.

As far as his colleague Samuelson was concerned, these policy-oriented models showed Solow to be like a "businessman on holiday", all "rough and ready". Samuelson preferred Solow in his habit as the "orthodox priest of the MIT school".[52]

In fact, as we shall see below, the Solow model of capital augmented technical progress, which made growth depend on the rate of saving while maintaining the marginal productivity theory of income distribution, was an early precursor of the growth models of the new classical economists. These early vintage models have been virtually ignored in the orthodox economics literature. On the one hand, these capital-augmented models of technical progress were inconsistent mathematically with the neoclassical requirement of balanced growth. On the other hand, the models were not formalized within a dynamic, optimizing, general equilibrium system. Because these vintage models at the time did not fit into any ongoing, coordinated research program, the models received little sponsorship.

IV. NeoWalrasian Growth Theory

A. The Golden Rule

The 1960s saw the heyday of national economic planning in developed and underdeveloped contries. Many economists saw that

> one of the basic problems in economic planning, in particular in underdeveloped countries, [was] concerned with the rate at which society should save out of current income to achieve maximum growth

(Uzawa, 1965, p.1). In the context of general equilibrium theory, this seemingly mundane assessment rationalized the revival of the highly formal, Ramsey model of optimal saving.

The development of the Ramsey model followed upon the introduction of the concept of balanced growth in economics and the popularization of variational calculus by physical control theory. Economists applied the fashionable Hamiltonian formalisms to model unending growth, constrained by the condition that utility per worker be conserved.

The mathematical innovations in capital theory in this period were conducted in an environment of metaphor and parody. Orthodox theorists referred to the state of balanced growth as the golden

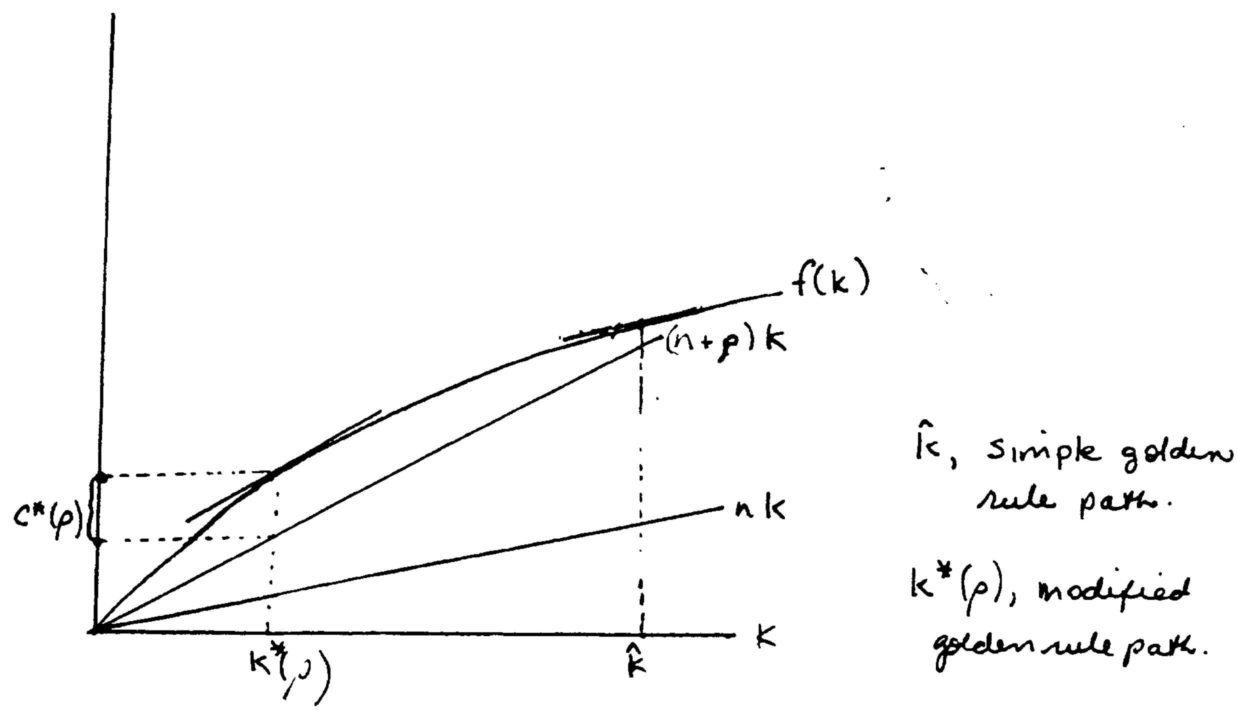


Figure 4.

The Golden Rule Path

(Koopman's 1963 Figure 8)

> Joyously, the Solovians hurried to compute the golden-age path,

the young Phelps enthused (1961, p.643).

> The year 1962-63 was a golden-year for Golden Rules at MIT,

Samuelson gloried (1965, p.487 n.4). General equilibrium theorists possessed a concept with which to forge their research in the direction of growth.

The basic optimal balanced growth model was presented by Koopmans (1963). Cass (1963) restated this model using the so-called Pontryagin-Hamiltonian formalisms.

Koopmans was a former physicist who pioneered structural econometric modelling and activity analysis. He gave his seminal paper on optimal growth during a study week on econometrics and development planning.

In the context of general equilibrium theory, the Koopmans seminar paper made two important contributions: (i.) the paper showed that the mathematical solution to the Ramsey problem of utility maximization was the same for balanced growth as for the stationary state with zero growth of the labor force; (ii.) the paper introduced shadow prices into the Ramsey model.

Koopmans stated the Ramsey problem,[54]

(21.) $_{\max}J(\sigma) = \int_0^\infty e^{-\sigma t}u(c_t)\,dt, \qquad \sigma = \rho - n, \ \rho > n.$

This asked for the maximum present value of the instantaneous utility of consumption per worker over the infinite horizon. The social rate of time preference was assumed to exceed the rate of growth of the labor force to ensure that that the integral converged. Koopmans then defined marginal utility as a commodity, with a present value of p_t,

(22a.) $p_t \equiv e^{-\rho t}u'(c)$.

Since the employment of capital made consumption possible, Koopmans defined the present value of the marginal product of capital in terms of marginal utility. Beforehand, in order that the model yield a solution consistent with the golden rule of growth, he defined the variable $f(k)$, which stood for output per worker minus the investment per worker sufficient to maintain a fully employed labor force,[55]

(22b.) $f(k) = f(k) - nk = c + \dot{k}$.

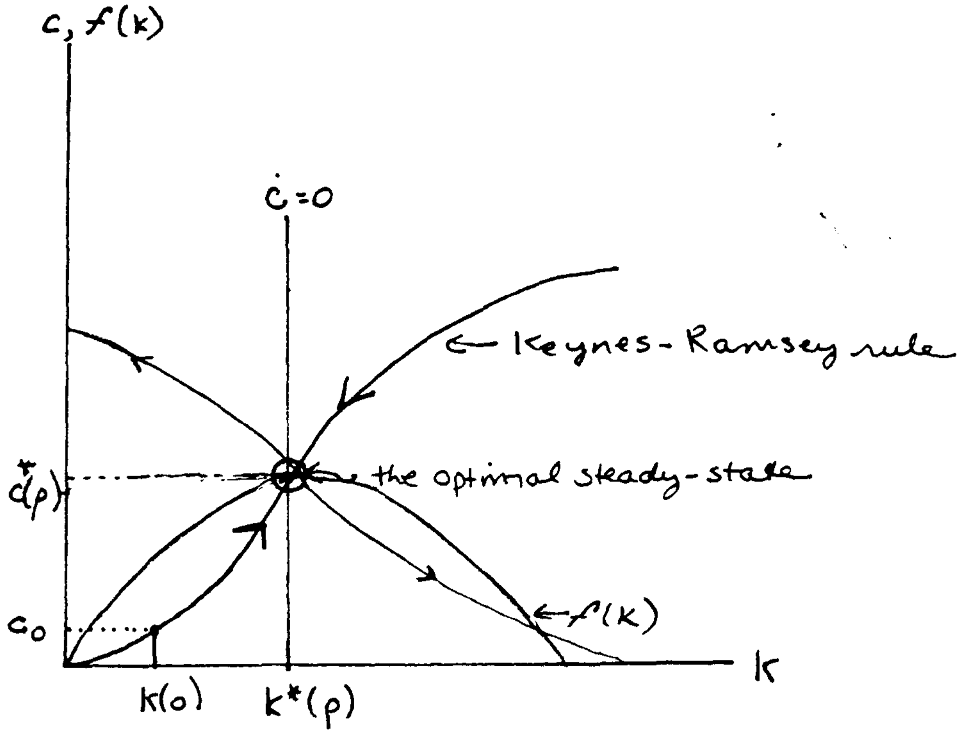
The price of the marginal product of capital in present value terms was

(22c.) $v_t \equiv p_t f'(k_t)$.

Utility was conserved along the unique, optimal path of the economy, given the satisfaction of the differential equation,

(22d.) $\dot{p}_t + v_t = 0$.

This equation, which was formally analogous to the continuity equation in the dynamics of fluids, was equivalent to the Keynes-



The optimal point.

(Koopmans, 1963, Figure 17).

Figure 5.

Ramsey rule.[56] Expanding equation 22d, given the definitions of p and v, yielded,

(22e.) $u'(c_t)(f'(k_t)-\rho) + u''(c_t)\dot{c} = 0, \quad t \geq 0.$

This meant that along the optimal path (i.) the proportionate fall in the marginal utility of consumption equals the excess of the marginal product of capital ($f'k$) over the rate of time preference; the marginal product of capital equalled the social rate of discount plus the rate of growth of the labor force.[57] Koopmans traced the approach to the golden rule path on the phase diagram shown in figure 5. The bold line showed the economy approaching this path given any initial capital-labor ratio.

The Koopman model showed how to regulate the rate of economic growth so that the economy approached the steady-state which maximized consumption per head, on the assumption of a diminishing returns technology. The social planner's job was to (i.) find the optimal initial level of consumption, given the capital-labor ratio and (ii.) in the absence of perfect futures markets, each instant to subtract from output that consumption level so that the excess of the marginal product of capital over social rate of discount completely offset the proportional fall in the marginal utility of consumption.[58] Then the planner knew the amount to be invested and repeated the calculation given the increased capital stock of the next instant.

After Koopmans gave the seminar paper, the members of the seminar, including Dorfman, Malinvaud, Morishima and Pasinetti, commented critically on a number of technical points. Nobody stepped outside the confines of the optimizing model that generated a smooth, steady trajectory of capital. The evidence that severe instability was a rare phenomenon and economies (at least the developed economies) experienced long run growth, supported this assumption.[59] The acceptance of the capital theoretic model carried unfortunate consequences for the future of economics. Growth and development subsequently became two distinct subfields. Growth economics offered little analytic basis for planning. No planner would have the information to maximize the social utility function. The dynamics of the optimizing model were deterministic, so that the optimal trajectory ran toward the future or the past. The optimal, steady-state path of the economy was represented by a geometric point in Euclidean space which represented a series of static equilibrium, indexed by time. Economic development, however, was a path and time-dependent process.

B. Economics and the Pontryagin Hamiltonian

The Hamiltonian formalisms offered an obvious way to formalize a dynamic optimizing model like Koopmans.

Economists who work with dynamic optimization techniques know that Pontryagin brought Hamiltonian calculus to the attention of

(23.) $J = \int_{t_0}^{t_1} f(x, w; t)\, dt$

was minimized. w stood for control variables -- fuel, temperature, voltage, etc. -- and was subject to constraints on its range. The controller chose w for each t so that when the position of the state variable x(t) was determined from

$$\dot{x} = h(x, w; t)$$

and the initial condition x(0) was given, the functional J was at a minimum.

Pontryagin and his colleagues defined the Hamiltonian,

(24.) $H(\Psi, x, w; t) = \Psi_o(x, w; t) + \Sigma_v^{n=1} \Psi_v f^v(x, w; t)$,

where Ψ_o was a constant. Equation 24 mimicked the Hamiltonian structure since the multiplier Ψ_v was defined as $\partial L/\partial \dot{x}$.[63] It meant that, given the initial optimal position of the state variable x(0), the controller would choose w to maximize H. The relation between the initial state (0) and the subsequent state was given by the "solution" Ψ_v. The equations of motion took the standard Hamiltonian form,

(25a.) $\dot{x}_i = \partial H/\partial \Psi_i$, $\quad i = 0 \ldots n$,

and

(25b.) $\dot{\Psi}_i = -\partial H/\partial x_i$.

The latter equation was modified if the optimal trajectory coincided with the boundary of the system. In this case, the Hamiltonian function took on a value of zero and the system was a conservative one. Finally, the "maximum principle" stated that

(26.) $H(\Psi, x, w) = M(\Psi, x)$,

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that is, at each instant, given Ψ and x, M() gave the maximum of the values of H(), as a function of the control variable w. This principle reduced the optimal control problem to the nineteenth century Hamiltonian calculus.[64]

Cass (1963/1965), who had just received his Ph.D. from Stanford University, and Uzawa (1965) were among the first economists to apply the control theory to dynamic optimization.[65] In order to characterize the optimal path of capital, they, as Cass put it,

> appeal(ed) to the general formulation of the classical calculus of variations developed by Pontryagin and co-workers, especially theorem 7, p.69 and the discussions on pp.189-191, 298-300

(p.234). Theorem 7 essentially restated the maximum principle. Pages 189-191 discussed the theoretical grounds for treating optimization during an infinite time horizon like finite optimization. Pages 298-300 stated the modified differential equation that an optimal trajectory on the boundary of the region must satisfy.

Samuelson in Foundations (1947) remarked that classical calculus could not maximize on the boundary or handle discontinuities.[66] The mathematical programming developed in the 1950s remedied the difficulty of dealing with constraints in the case of economic statics (the Kuhn-Tucker (1950) theorem dealt with nonlinear solution) One of the Ph.D. students under Bliss in the 1930s dealt with this problem in variational calculus, but the papers remained unpublished.[67] Thus the issue seemed largely unresolved in economic dynamics. This provided the technical rationale for the interest of economists in the work of Pontryagin et al.[68]

Cass defined the problem of the central planning board, to find the growth path to maximize the welfare functional

(27a.) $\quad {}_{\max}J(\sigma) = \int_0^\infty u(c(t))e^{-\sigma t}dt, \qquad u'>0,\ u''<0,$

s.t.

$$y(t) = f(k(t)), \qquad f'(k)>0,\ f''(k)<0.$$

k stood for investment per worker, defined as

(27b.) $$k(t) = \dot{k}(t) + nk(t),$$

$$k(t) \equiv y(t) - c(t), \qquad c(t)\geq 0, \quad z(t)\geq 0.$$

The inequality constraints meant that all of output could go to consumption or investment. $\dot{k}$ was investment over and above that required to maintain the full employment capital-labor ratio. When k=nk(t), the capital-labor ratio was constant.

The Hamiltonian function was

(28.) $$H = u(c) + p(f(k)-nk) \qquad p=u'(c).$$

u(c) stood for instantaneous utility and the constraint, $f(k)-nk$, for net investment per worker. p was defined as the marginal utility of a unit of investment (a definition that followed from $pH/pk = 0$).

Table 3 summarizes the analogies implied by the borrowing of control theory of economic growth theory.

Table 3

The Formal Analogies

Economics		Pontryagin et al	
instant. utility	$u(c)$	$\Psi_o(x, w, t)$	initial state
constant	1	Ψ_o	constant
investment	$\dot{k}$	$f^v(x(t))$	trajectory
shadow price	p	Ψ_v	solution
allocation	c, k, s per head	w	controls

However, in contrast to Pontryagin and his associates, the economists explicitly did not think out the Hamiltonian in terms of its underlying (Lagrangian) structure.[69] Yet, the mathematical solution to the optimal economic path depended on the underpinnings that economists assigned the Hamiltonian.

Cass interpreted the multiplier p in terms of a shadow price, or marginal utility. This interpretation was not entirely new. It was conventional to define the Lagrange multiplier in static, constrained maximization problems, as a competitive market price that reflected marginal utility. Samuelson and Solow commented in the 1956 essay that

> (i)t would ... be possible to give price interpretations to the Hamiltonian momenta

$[p_i(t)]$,

though they neglected to do so (p.561).

The definition of the shadow price as the measure of marginal utility was crucial to the economic interpretation of the Hamiltonian.[70] It meant that, given the initial capital stock and price of capital, the level of investment was consistent with the equality of the indirect marginal benefit p of an increase in investment and the immediate marginal opportunity cost u'(c). That p=u'(c) determined the position of the system at the next instant, and so on, for the whole trajectory.[71] This reasoning, of course, came down to the Keynes-Ramsey rule based on marginalist reasoning.

Compared to the days of Walras, by the 1960s, economists were better equipped to render marginalist principles in terms of "the (classical) law of energy". Just as energy was conserved in classical mechanics, so the imputed present value of the net national product took on a constant value. Just as the physical system chose a path x(t) to minimize the total energy H,

$$H = -L + \Sigma p\dot{x}$$

of that system, so the economic system would choose that vector of investment in

$$H = -L + \Sigma pk$$

to maximize total utility.[72]

Cass introduced the "canonical equation" that had to be satisfied for the path of capital to optimal. Traditionally, the continuous imputed price was defined by the change in the

Hamiltonian function in respect to investment,

(29a.) $\dot{p} = -\partial H/\partial dk, \quad \dot{k}=0.$

Cass put this in present value terms,

(29b.) $\dot{p} = -\partial H/\partial k \; \{e^{(-\rho)t-to}\}, \qquad p = pe^{\rho t},$

$= -u'(c)(f'(k)-n) + \rho p.$

This differential equation implied that, along the steady state optimal path, where the price of capital per worker is constant, that is, $\dot{p} = 0$, the marginal product of capital equals the social time rate of preference plus the rate of growth of the labor force (the golden rule).[73] (Along the optimal path, the limiting present value ($t\to\infty$) of the capital stock equalled zero, which closed the system.) In current value terms, the differential equation read,

(29c.) $\dot{p} = -\partial H/\partial k + p\rho,$

which was the modified "canonical" form that economists would use.

What did all this have to do with Pontryagin and the maximum principle? For the capital theorist of the 1960s, the maximum principle meant that

> (i)f the trajectory of an economic system is determined by the condition that its Present Value attain a maximum, then the decision *k* must be chosen at instant t during the time-interval (0,T) so as to maximise the (flow of) total imputed value [39] generated by the system at that instant

(Magill, 1970, p.55). This rendered the maximum principle as (i.) a helpful notation and (ii.) a problem of maximizing present value. The notation originally was broached in applications of variational calculus in economic planning in Europe in the 1940s, but was fashionable in the US until the 1960s, when economists there lent it a marginalist interpretation.[74] The main point is that economists did not need physical control theory to transform the Hamiltonian from present value to current value. A little algebraic manipulation, as shown above, would do.[75]

The Hamiltonian functions used in economics and the engineering were indeed two different functions: The physical control problem was defined in terms of three variables, position (x), velocity ($\dot{x}$), and the control (w); the economic problem in terms of two variables, position (k) and velocity ($\dot{k}$), according to the classical calculus of variations.

The technical contribution of Pontryagin and his associates in economics concerned the treatment of maximization subject to constraints in the form of inequalities. Just as physical controls were subject to constraints, so the propensity to save was constrained, between the values of zero and one.[76] The system "jumped" when the value of the propensity to save changed from a limiting to an interior value. To this extent, economists thought of control theory as simply a "'Kuhn-Tucker' type generalization of the traditional canonical form to the case in which the 'static' maximum is attained on a boundary rather than at an interior point of the output space" (Burmeister and Dobell, 1970, p.370).[77] But this whole fuss over boundary maxima concerned a mathematical nicety. In practice, a country's propensity to save was never zero or one.

American economists in the 1960s were impressed by the recent source of variational calculus, prompted, as it was "by the requirements of space technology" (Dorfman, 1969, p.817). Before a seemingly arcane subject, the calculus of variations in the 1960s becmae known as optimal control theory, giving capital theory "a new lease on life".[78]

Burmeister and Dobell claimed that "Academician Pontryagin and his colleagues have thus enunciated a newer and more powerful principle of an invisible hand; the maximum principle of Pontryagin is seen to be the culmination of a logical sequence originating in the maximum principle of Adam Smith" (p.404). But the problem of Pontryagin et al concerned the use of <u>exogenous</u> <u>controls</u> on a system.

In Walrasian economics, the instantaneous allocation of resources to investment and consumption served as an implicit instrument of control. Given perfect futures markets, the economic system traded off the size of the capital stock with its rate of change, which is what makes the stock change in size, so that consumption over time is maximized and utility is conserved. Economic planners have only to assign the price of investment that

maximizes the net national product given the initial capital-labor

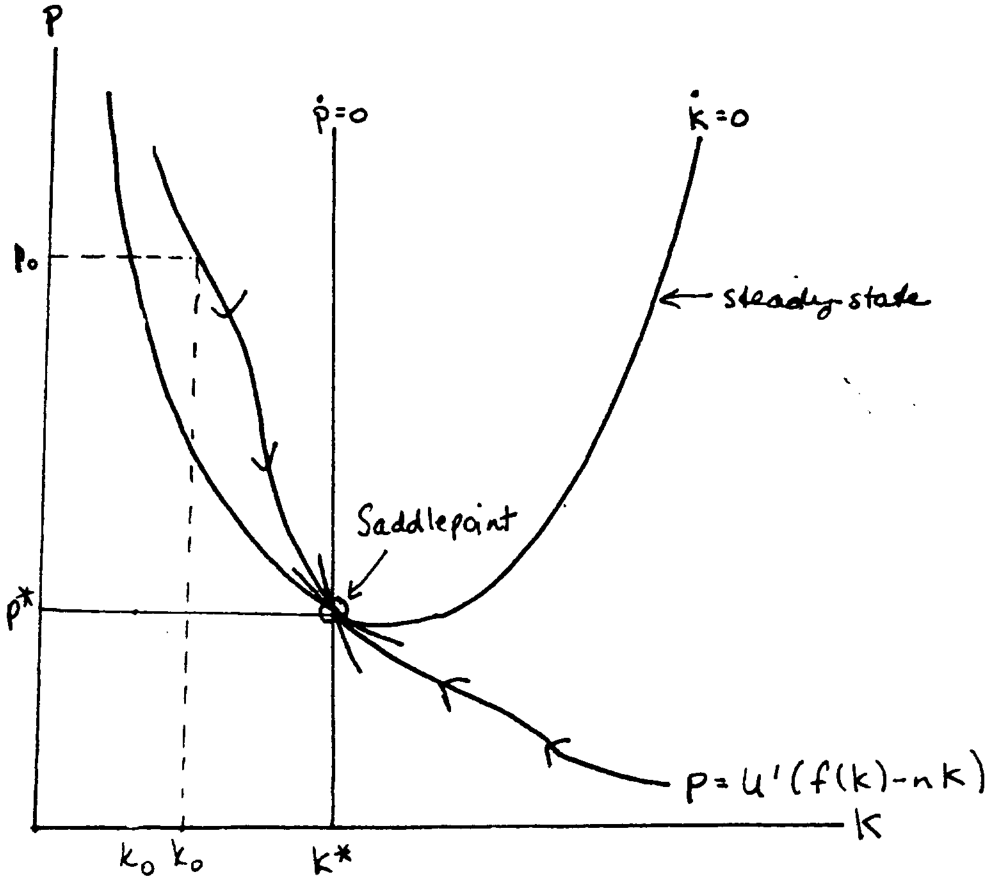

Figure 6.

The optimal growth path, $p^* k^*$.

(Cass 1963/65 Figure 1)

ratio and reassign corrected prices in the event of shocks to the system at the unsteady saddlepoint equilibrium, shown in Figure 6.[79] Classical variational calculus, rather than the formalisms of control theory, was the appropriate medium (and the form that economists actually used) to express this problem.

Textbooks in capital theory still refer to Pontryagin to legitimize the dynamic competitive equilibrium system with perfect foresight and continuous asset market clearing. This analogy is largely pretense.

V. An Empirical Growth Theory: Kaldor's Model

A. Solow vs Kaldor

Solow, having just won the Nobel Prize for founding modern growth theory, remarked at the AEA meetings that growth was about once again to become the topic of the day. In the preceding fifteen years, productivity growth in most of the OECD countries was negligible and many countries, including the US and the UK, saw negative growth rates, yet the economies of the "Gang of Four" experienced double-digit export-led growth rates and Japanese exports accounting for a large part of a growing deficit on the US balance of trade. Growth ranked high on the agenda of economists interested in policy issues. Meanwhile, new classical economists, the new creative theorists of economics, reached a stage in the course of their research where growth was an obvious research problem. We find the new classical economists demanding, using the jargon of the 1950's, "What are the engines of growth?" (Rebelo (1987), p.2). In the growth models presented by new classical economists, the economy, which is peopled by agents who have perfect foresight and maximize utility, follows an optimal path, along the lines set out by Cass (1963).

At the same time, these new growth models are intended to explain the stylized facts of growth, which were established by Kaldor in the 1950s. As the new classical economist Romer explained, "the basic questions about growth are being reexamined -- it may be useful to review ... Kaldor's list of facts" (1989, p.54).

Kaldor, a well known economist at Cambridge, England in the 1950s, insisted that economists start with a "'stylized' -- i.e. non-rigorous but suggestive -- description of a modern economy" (Samuelson, 1963, p. 197). His stylized facts of growth included the existence of:

(1.) continued growth of labor productivity, with no tendency for a falling rate of productivity growth;

(2.) a positive correlation between export growth and productivity growth.

(3.) a continued increase in the capital-labor ratio;

(4.) steady capital-output ratios (however capital may be measured), or at least the absence of clear long term trend in the positive or negative direction, once less than full capacity utilization was taken into account;

(5.) a constant rate of profit on capital (Economists have questioned the degree of confirmation of this fact, but it remains accepted as a broad generalization by orthodox economists.)

(6.) a constant share of investment in output and profits in income (a fact that followed from facts #4 and #5);

(7.) no tendency for country growth rates to converge (Kaldor, 1958, 1966a).

Kaldor claimed that Solow's growth model could explain these facts. The artifice of exogenous technical progress in the Solow model generated, but did not explain the existence of productivity growth (facts #1, 2). In the Solow model, which assumed diminishing returns, an increase in the capital-labor ratio (fact #3) involved a <u>rise</u> in the capital-output ratio and a <u>fall</u> in the rate of profit and the share of profit in income, outcomes which conflicted with the stylized facts (#4,5,6). The Solow model predicted that countries' growth rates tend to converge (in conflict with fact #7). Assuming identical production function, perfect capital markets and complete mobility, capital flowed from rich to poor countries, where the rate of profit on capital was highest. This tendency to convergence was reinforced by the diffusion of technical progress from rich to poor countries.

In fact, the new classical growth economists of the 1980s made the same criticisms of the Solow model as Kaldor. Given their shared criticisms of neoclassical growth economics, the Kaldor and the new growth models are similar. Both models attempt to endogenize growth and assume increasing returns. In both models, long run growth is driven by the accumulation of embodied knowledge by forward-looking, profit seeking agents. Indeed, the new classical criticisms of the neoclassical model reflects the old, Kaldor-Solow controversy over growth via the medium of an optimizing model.

The debate over growth in economics initially began in the 1950s, just when growth became an issue in policy circles. Growth rates in the developed countries were remarkably high. The media reported that Japan was growing by 10 percent per annum, the Soviet Union by 7 percent a year. Western policy-makers, who just learned about demand management, wanted to achieve comparaable rates of growth. Meanwhile, Kaldor and Solow were embroiled in the Cambridge-Cambridge controversy over how to model production and growth. Both of these economists saw that technical progress provided the main explanation for the rapid growth of production, but their models of technical progress differed.

In Kaldor's theory, which assumed "full employment", technical progress was embodied in capital and, at any time, the growth of demand for capital would induce technical progress. In neoclassical parlance, it was as if movements *along* the neoclassical production function led to a *shift* in that function. There were, in other words, increasing returns. According to Kaldor,

> "allowing for increasing returns was sufficient to cause the whole structure to collapse like pack of cards. It is high time that the brilliant minds of MIT were set to evolve a system of non-Euclidean economics ... where such abstractions are initially unnecessary"

(1966b, p.297). In light of his theory, Kaldor argued that demand management that promoted capital investment would increase the rate of technical progress and the "potential" rate of growth.

Orthodox economists in the US have not thought highly of Kaldor's growth theory. In 1989, a conference was held on economic methodology that included a paper on new classical economic growth theory.[80] In the discussion of this paper, Hal Varian, an orthodox economist, explained the recent background to the new classical growth theory and emphasized that its development was prompted by current economic events. A member of the audience asked Varian what he thought of Kaldor's growth economics. Varian shrugged and replied, "Kaldor had no theory". Similarly, Romer (1989) dismissed Kaldor as a theorist because he lacked "a tractable alternative model".[81]

Kaldor attained prominence in the interwar period for his contributions to mainstream economic theory.[82] He knew little differential calculus, Samuelson (1988) commented, but, when confronted with a model of the trade cycle or production, he saw "how it had to go" (p.321). After World War II, Kaldor's research changed tack. Its aim was to develop an alternative model to the one used by mainstream economists.

In a discipline that was becoming increasing mathematical, Kaldor's lack of mathematical sense was a drawback. His mathematics ressembled Keynes' in style, though Kaldor lacked Keynes' training in the field. Most of Kaldor's mathematics was algebra and each algebraic term had an observational counterpart. His mathematical arguments started off with an identity, which he elaborated into a causal relation, in the same way that Keynes's investment multiplier came from the GNP accounting identity. Most of his models did not solve for anything, but instead were a series of derivations. Even with the help on various occasions of Champernowne,[83] Mirrlees,[84] Hahn and Pasinetti,[85] Kaldor failed to anticipate technical problems, which upon occasion got him into serious trouble.

Once Kaldor set out to criticize mainstream economics, he went about it like a bull in a china shop. Wishing to make an independent attack on orthodox economics, Kaldor focused on a problem with which the mathematical economists at the time were not equipped to deal, the problem of increasing returns.

1. Externalities and Equilibrium

Romer (1983) in this Chicago Ph.D. thesis introduced the idea of increasing returns as "as old as the attack on the ideas of Malthus" (p.9). The reference is rather obscure, but idea of increasing returns indeed had a turbulent history.

The idea of increasing returns originated in the first three chapters of Adam Smith's *Wealth of Nations* (1776). There Smith discussed the scale economies that allowed increased specialization both within the factory and between trades as a result of learning-by-doing and indivisibilities. As Romer (1986b) narrated,

> Adam Smith, took the view that the degree of specialization at any point in time was limited by the extent of the market. He went on to suggest ... [that] increases in income lead to increases in demand, which can in turn lead to increases in the extent of the market. These permit increases in specialization, which permit output to grow faster than inputs, so per capita income rises. Repeated in circular fashion, this argument appears to generate a process of unending economic growth.

Marshall's Principles (1896) developed the idea of increasing returns. He defined two types of scale economy. Internal economies depended on the resources of the firm. External economies depended on the general development of industry and included, for example, aggomeration economies and trade knowledge. Any firm's supply curve depended on both internal and external economies, so that marginal cost depended on both scale and time, with cost reductions irreversible because learning was involved. Hence, Marshall considered that, "the high theme of economic progress ... [showed] the insufficiency of the statical method".[86]

In the 1920s, a debate occurred over the implications of Marshall's discussion of increasing returns. One of the participants in the debate, Marshall's protegé Pigou accepted the notion of external economies, but attempted to keep the statical analysis (Pigou, 1927). Pigou identified external economies at the level of the competitive industry, which was composed of Marshallian competitive firms, which chose a quantity and then set the price, a process akin to monopolistic competition. External economies continuously, slowly and imperceptively reduced the costs of the firms in the relevant industries. Pigou proposed a subsidy on these industries so that prices equalled true marginal costs and the level of output was socially optimal.

Kaldor learned about increasing returns from his tutor, Allyn Young, a former Harvard Professor and fellow at the National Bureau of Economic Research who became head of the economics department at the LSE in the late 1920s. Young, an avid reader of Marshall's *Principles*, contributed to the continuing debate about increasing returns.[87] He thought that Marshall's concept of external economies was useful because it explained why the presence of increasing returns did not lead to monopoly. There were two sorts of external economies, which reduced costs: (i.) Internal economies of firms producing inputs figured as external economies from the point of view of firms buying these inputs. The presence of these external economies, Young suggested, did not hinder the establishment of a stable, static equilibrium. (ii.) The creation of new products and new industries also acted as external economies. Young analyzed these economies using Marshall's concept of elasticity. Increasing returns meant that the continued increase in the production of the goods of any one industry imposed a proportionately smaller opportunity cost in terms of other industries. If two industries produced with increasing returns and demand for the goods of each industry were elastic, then the demand for either industry would be the reciprocal of the supply of the other. In these conditions, Young stated,

> (n)o analysis of the forces making for economic equilibrium, forces which we might say are tangential at any moment of time, will serve to illumine the field, for movements away from equilibrium, departures from previous trends, are characteristic of it.[88]

As Young lectured to his students (who included Kaldor),

> (s)eeking [static] equilibrium conditions under increasing returns is as good as looking

> for a mare's nest.[89]

The equilibrium in question was a dynamic equilibrium which could be studied only at the macro-level, Young emphasized.

Young appended a note to the 1928 essay, in which he changed tact. The note, designed to persuade welfare economists to incorporate increasing returns into their analysis, stated the now well established argument that increasing returns were consistent with a static equilibrium, as long as the production possibilities curve was less sharply convex than the indifference curve.[90]

Perhaps it was this appendix that has led the new classical economist Romer to miss the main point of Young's essay about external economies and disequilibrium. According to Romer's Ph.D. thesis (1983),

> in an article published in 1928, Allyn Young gave a verbal discussion of economic growth driven by the increasing returns resulting from specialization. Much of this discussion is problematical, but he seems to understood unbounded growth, that externalities could permit the existence of a competitive equilibrium.[91]

And later, Romer insisted that

> Young ... following Adam Smith ... appears to generate a process of unending economic growth. Moreover, Young argued, the increasing returns due to specialization could

> be view as being external to any individual firm in the sense proposed by Alfred Marshall, so this explanation of growth could be consistent with the existence of a decentralized, competitive equilibrium.[92]

Romer subsequently devised what he called the "Marshall-Young-Romer model" of production with externalities. This model pertained to the static equilibrium at the level of the firm.

The externalities that Romer had in mind were technological externalities, as defined by Scitovsky (1951), Meade (1952), Arrow (1962) and Chipman (1970). They assumed the existence of identical, small, competitive firms, in the absence of the Walrasian auctioneer. The technological externalities involved a parameterization of the production function, so that the output of the firm depended on its own inputs and the outputs and inputs of other firms. Each firm knew that its productivity depended on the aggregate scale of output, but because each firm was small, they neglected their contribution to aggregate output. Since technological externalities involved the direct interdependence between firms, it had no affect on the establishment of a unique general equilibrium. However, this general equilibrium was Pareto sub-optimal, since firms' marginal costs diverged from social marginal costs. This outcome called for taxation or a change in the institutional setting.

Kaldor had no interest in the problem of fitting external economies into the static, marginalist framework. He extracted that part of Young's discussion of increasing returns that Young took from Marshall and Marshall took from Smith -- which Romer, educated in the Walrasian tradition, later would dismiss as "problematical".

2. Kaldor's model of increasing returns

With great originality, Kaldor elaborated a theory of increasing returns in order to explain his stylized facts.[93]

Kaldor derived his theoretical model of increasing returns as follows. He assumed a constant capital-output ratio, so that

(30). $\Delta K/K = \Delta Y/Y$,

the rate of growth equalled the growth of output.

Next, he defined that

(31.) $I \equiv \Delta K$.

Substituting this identity into equation 30, dividing by Y/Y and rearranging gave the Harrod Domar equation for the natural rate of growth,

(32.) $I/Y = (\Delta Y/Y)/(K/Y)$.

Kaldor expressed the left hand side in terms of investment (rather than saving) since in Keynesian theory, investment was the exogenous variable that caused output to rise so that in equilibrium savings equalled investment.[94]

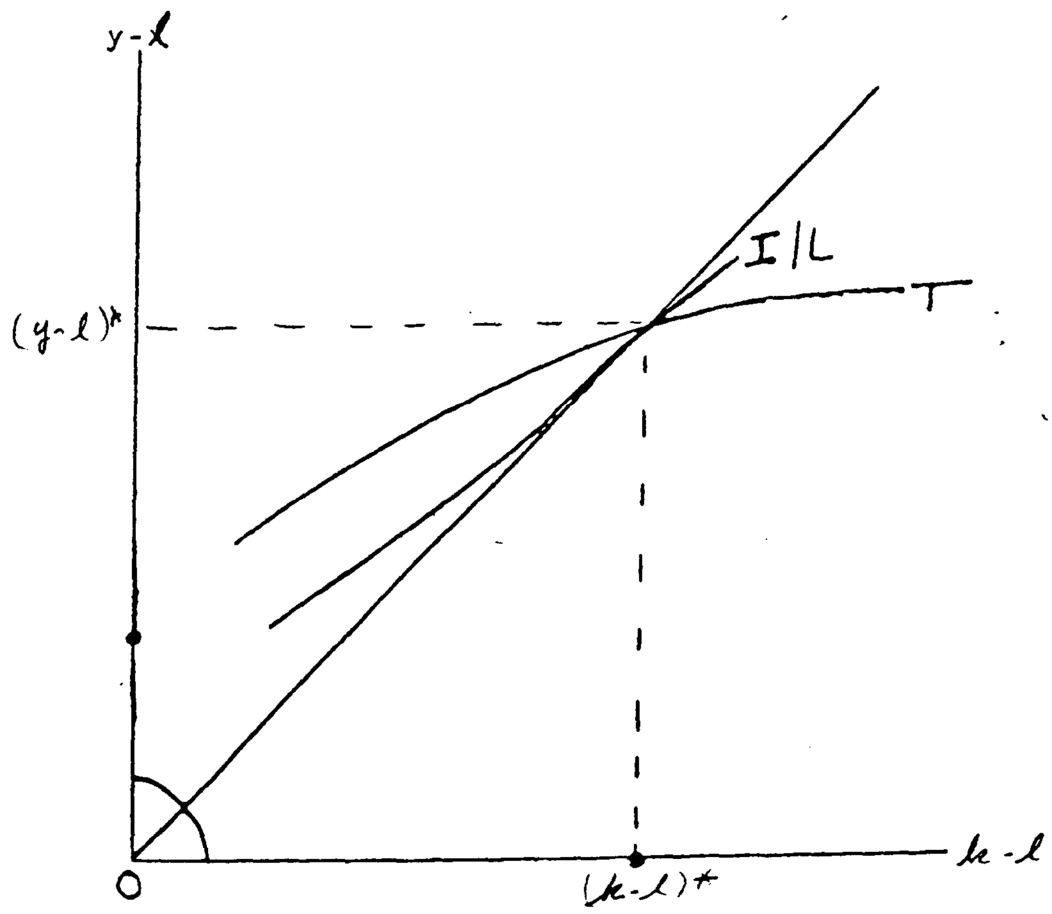

Figure 7. The Interaction of the Technical Progress Function (T) and the Inducement to Invest Function (I/L).

(Kaldor, 1957, Figure 1).

Until firms found their long run equilibrium growth level, the rate of profit would vary, so that

(33.) $I/Y = (\Delta Y/Y)/(K/Y) + \beta\ \Delta\pi/\Delta K.$

This equation in per capita terms was represented by Figure 7.

This Figure took the conventional form of the geometry of growth economics in the 1960s.[95] Thus the geometrical form of this Figure was identical to Solow's Figure (2, above), but the axes bore different labels.

In Kaldor's figure, the vertical axis stood for productivity growth and the horizontal axis for the growth of capital per worker. When the growth of capital per worker was less than the equilibrium growth rate (*), productivity grew faster than capital per worker and the rate of profit on capital increased. Conversely, the rate of profit fell when capital grew faster than the equilibrium rate. Given profit-seeking, oligopolistic firms, the rate of investment per worker (along I/L) tended to the level (*) at which the rate of profit was maximized. At this long run equilibrium level, productivity growth equalled the growth of capital per worker and the rate of profit and the capital-output ratio were constant. The coordinate $(y-\ell)^*(k-\ell)^*$ described a position steady growth and explained facts #1-5.

The nonlinear curve T in figure 7 stood for the technical progress function, which Kaldor stated for convenience in the linear form

(34.) $P = r' + n\ (k'-\ell)$, $\ell=\Delta L/L$

$P=(\Delta Y/Y)-(\Delta L/L)$

$k'=\Delta K/K$

This meant that the growth of productivity (P) was associated with the rate of growth of capital per worker. In other words, there were increasing returns. As Figure 7 shows, technical progress was *not constant* and was *determined endogenously*, by the rate of economic growth and investment. At any stage of a society's development, technical progress exhibited an upper bound. The technical progress function shifted up (never down, because progress was irreversible) given a change in the technical dynamism of a society.

Kaldor intended the technical progress function to replace Solow the production function, which made economic growth a unique function of the growth of factor supplies, given a residual due to exogenous technical progress. Kaldor argued that Solow (1957) used circular reason when he identified technical progress as a residual: From the outset, Solow, normalizing for labor, identified the the share of profits in income by the <u>slope</u> of the production function. This meant that the <u>shift</u> in the curve <u>must</u> be due to a residual entity that did not receive a factor reward. In contrast, in Kaldor's model, investment, leading to increases in the capital-labor ratio, caused the production function itself to shift. Thus Kaldor found "no conceivable operation by which the slope of this 'curve' could be identified" (Kaldor, 1958, p.206).

Kaldor's case was less clear cut than his debunking of the production function made it out to be. It was obvious to his neoclassical audience that the technical progress function (equation 39) was the same as a Cobb Douglas production function, normalized for labor and differentiated totally in respect to time. But the linear technical progress function could not be integrated to obtain the Cobb Douglas function, because the constant of integration depended on the initial conditions, the distribution of capital and labor between industries and the age-distribution of the capital stock.[96] In other words, in Kaldor's model, progress was path-dependent. In contrast, the Cobb-Douglas production function was independent of initial conditions.

V. The New Growth Models

The growth models of the 1980s, like the Kaldor model, are designed to incorporate increasing returns and endogenize growth, replace the neoclassical model of exogenous technical progress, and explain the stylized facts of growth. The leading proponents of the new growth theory are new classical economists, including Lucas; his student, Paul Romer; and Romer's student, S. Rebelo.

Facts, it is often said, are made, not given.

The neoclassical model predicted that countries income levels will converge to a common growth path, on the grounds that technical know-how diffused and capital flowed from rich to poor countries. Economists in the 1970s tested this prediction by regressing country's per capita growth rates on their per capita income, which served as a proxy for the level of technology. The tests resulted in a *negative* regression coefficient, which confirmed the theory.[97] With the development of new classical economic growth models, these statistical results became open to question.

Baumol (1986) used the Maddison sample to show that the growth rates of the sixteen richest countries today converged from disparate levels in 1870. Romer (1986) responded that an *ex ante* sample, that is, a sample of growth rates of the richest countries in 1870, might give a different result. de Long (1988) followed up Romer's suggestion. Using the Maddison sample, he showed that the growth rates of the richest twenty-two countries in *1870* had not converged over the ensuing century.

In response to de Long, Baumol and Wolff (1988) took an *ex ante* sample off the Summers-Heston data for 124 countries, 1950-1980. They showed that the growth rates of the richest fifteen countries converged. Romer (1989) then replicated the Baumol and Wolff study. Using a sample of 115 countries, which involved a lower quality of data, he showed that growth rates during 1960-1980 showed no tendency to converge. Taking all countries together, Lucas concluded, the correlation between their "income levels and rates of growth ... would not be far from zero" (1988, p.4).

In sum, the debate over the existence of convergence vs nonconvergence rested on sample selection bias. Economists found samples to confirm their competing views of the facts.

Nevertheless, the new classical growth literature has treated the facts of growth as given. The new classical economists have said their contribution is to replicate, account for, and offer an alternative interpretation of the facts, on the supposition that such a redescription increases understanding and amounts to a discovery.

B. What is New Classical about the New Growth Models?

In the context, of orthodox economics, new classical economics represents a technical advance. The new classical school, as it developed the 1970s, treated the assumptions of the competitive equilibrium program -- perfect rationality, optimization and so on -- so seriously that exemplified the orthodox, Walrasian research program. But, while Walras (1900) invited economists to treat the competitive price mechanism as a branch of the mathematics of maximization, he did not know enough calculus to suggest mathematically why his price formulae solved the problem of maximization subject to constraints. The problem received little attention until the renewed interest in a general theory of equilibrium in the 1930s. Samuelson, Arrow, Debreu, Koopmans, and others then spent thirty years finding and applying the tools of vector and differential calculus to show the conditions in which the competitive price equilibrium existed mathematically and was stable at any point <u>in time</u>. The latter 1950s found Samuelson and the next generation of mathematical economists faced with the problem of showing the optimal <u>time-path</u> of the competitive equilibrium. The new classical growth models are in part the outcome of the solution of this mathematical problem.

The content of the new classical growth theory is far from novel. There is a large overlap between the developments in trade theory during the last decade and growth theory. Trade theorists challenged the neoclassical model based on comparative advantage and constant returns to scale. Increasing returns due to the division of labor and specialization became as important as comparative advantage as a source of trade: while comparative advantage explained why countries with different endowments traded, increasing returns could explain why similar countries traded. The problem was that increasing returns conflicted with the assumption of perfect competition. To solve this problem, trade theorists identified increasing returns as an <u>external effect</u>.[98] As Krugman explained, "(t)he traditional way to model trade in the presence of increasing returns has been to assume that these scale economies are external to the firm. This assumption has been historically favored because it allows one to avoid the problem of market structure: with external economies one can preserve the assumption of perfect competition" (1985, p.45). For this reason, the new growth models also has treated increasing returns as an external effect.

Since the new growth theory has relied heavily on the trade and externalities literature, does it matter that new classical economists have developed this growth theory? In some ways, yes, in others, no.

Yes, in four ways:

1. The new classical models are a logical outcome of the new classical research agenda.

New classical economics assumed that markets continuously clear. Business cycles, which in the neoclassical model represented deviations from an exogenous trend, in the new classical models reflected jumps of the trend due to exogenous shocks, which arose from technological change or government intervention. Instead of the neoclassical organization of the time series as a deterministic trend and a stochastic component, new classical economists modelled a stochastic trend and a transitory component. The transitory component arose while agents collected full information about shocks. Real business cycle theory in the 1970s studied the transitory component. The point of the new growth theory is to explain the trend.

2. That the new models were presented by Lucas and his associates attracted the interest of the economic establishment in growth. The growth models have appeared in unpublished papers circulated, privately, as departmental working papers or NBER conference papers, amongst orthodox economists working on optimal growth theory. Not all the papers have been available upon the request of economists at large.

3. The new growth models require mathematical expertise, which has characterizes the new classical school.

In the 1960s, Uzawa and others constructed growth models with increasing returns using the Hamiltonian formalism, but their methods were not standardized and the mathematics went over the heads of the vast mass of economists. In the 1970s, economists used Hamiltonians to formalize the optimal use of exhaustible resources.[99] Graduate economics courses became more mathematical and, at least at MIT and Chicago, included variational calculus, a course taken by undergraduates in physics. As Romer reflected, "in the years between 1970 and 1980, the discussion of the theory of aggregate consumption moved from a point where it would have been impolite to mention Euler equations to a point where it was impossible to carry on a discussion without them" (1989, p.52). It is not surprising that the recent crop of graduates who have excelled in applied mathematics should acquire an interest in new classical research.

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aims has dominated the new growth theory. Lucas has stressed the deficiencies of the neoclassical growth model. As he remarked,

> Solow's 1956 conclusion that changes in saving rates are level effects ... was startling at the time, and remains widely and very unfortunately neglected today. The influential ideas that changes in the tax structure that make savings more attractive can have large, sustained effects on an economy's growth rate sound so reasonable, ... but it is a clear implication of the [neoclassical] theory we have that it is not ... [Neoclassical] theory is not, as it stands, a useful theory of economic development: [witness] its apparent inability to account for observed diversity across countries ... I will begin by considering an alternative, or at least a complementary, engine of growth to the 'technological change' that serves this purpose in the Solow model

(1985/88, pp.12, 17).

No, in three ways:

1. The new classical real business cycle models assumed rational expectations, meaning that agents know the true

econometric model of the economy, up to a serially uncorrelated error. Rational expectations implied continuous market clearing, in the sense that, abstracting from white noise, every price was market-clearing for agents who know the price, with the result that agents successfully optimize. These two assumptions, of rational expectations and successful optimization, defined the new classical economic research project in the 1970s.

The new classical economists have relinguished these stringent assumptions in their growth models. Consumers and firms continue to have rational expectations about future prices, but in the deterministic context of long run growth, rational expectations, can mean nothing more than perfect foresight or the existence of perfect futures markets. As Lucas explained, "(f)or this particular [Hamiltonian] model, with convex preferences and technology and with no external effects of any kind, ... the *optimal program* is also the unique *competitive equilibrium* program, provided ... [that] consumers and firms have rational expectations about future prices. In this deterministic context, rational expectations just means perfect foresight" (1985/88, p.12).

2. New classical economists in the 1970s assumed continuous optimization. In the new growth models that attribute increasing returns to the external effects of capital accumulation, agents have perfect foresight only of private costs, which diverge from social costs, so that the path of the economy is Pareto sub-optimal.

3. Lucas charged that the neoclassical growth model cannot show the effects of government policy on growth. In light of the "Lucas critique" of the 1970s, which stated that policy interventions were ineffective, this charge comes as a complete turnabout.

In sum, the new classical economic research project of the 1970s, has proved to be unsustainable in modelling growth.

C. Increasing Returns, Externalities and the Cobb Douglas Production Function

Romer, as a Ph.D. student at the University of Chicago, recognized that

> increasing returns [to scale] is surely most controversial [topic]. A presumption in favor of diminishing [marginal] returns seems to persist despite repeated findings that the rate of growth of inputs cannot account for the rate of growth of output

(p.6). As Romer explained, increasing returns to scale seemed to conflict with the marginal productivity theory of income distribution. It is, he stated,

> mathematically impossible for all factors of production to be paid their marginal products. With increasing returns, this would more than exhaust total output. Models constructed

> during the 1960s resolved this by assuming that A [exogenous technical progress] came from the sky, or perhaps from the National Science Foundation, and therefore did not need to be compensated in the market

(p.99). Romer and Lucas avoided these problems by modelling increasing returns as the external effect of capital accumulation.

In particular, they amended the Cobb-Douglas production function by a multiplicative factor that represented a positive externality. Given this amendment, their production function was observationally equivalent to the Solow production function with exogenous technical progress. In both models, the growth of the factors, weighted by their shares in income, was insufficient to account for the growth of output. The difference between the neoclassical model of exogenous technical progress and the externalities model was that the assumption of externalities involved, in principle, a testable hypothesis about firms' behavior. On this basis, the new classical models, in contrast to the neoclassical model, proposed to explain productivity growth and the effect of taxation on growth.

<u>The traditional modelling gambit, the production function modified by an externality, when specified as the constraint in the dynamic optimization problem, offered the key to the solution of the new classical problem of growth.</u>

Lucas and Increasing Returns

Lucas modelled externalities to human capital accumulation (1985/88). The outcome of the model served to replicate the stylized facts of growth, inequality in countries' growth rates and the correlation between net exports and growth. In principle, the model was testable. But the main purpose of the mathematics was exploratory and conceptual: it was intended to direct readers to think about growth in a particular way. The new classical economists presented many different production functions, that constructed different aspects of growth. The (mathematical) media was the message.

Lucas stated the production function, introduced by Uzawa (1965),

(35.) $Y = AK(t)^{a}[w(t)h(t)L(t)]^{1-a}h_{a}(t)^{v}$, $0<w<1$, $0\leq h\leq\infty$, $a<1$,

where A stood for the level of technology; w, the fraction of work time that the devoted to the production sector; 1-w the fraction devoted to the education; and h, the internal effect of the level of human capital. A worker who chose to accumulate human capital did so because of the private benefits that this would bring. Because workers interacted, all workers actually benefitted from the additional training of any one worker. The external effect, $h_{a,}$ multiplied the productivity of each worker at any skill level. As a result, this model, Lucas stated, "exhibits sustained per-capita income growth from endogenous human capital accumulation alone: no external 'engine of growth' is required" (p.19). Lucas proposed a education subsidy to internalize the externality and increase the rate of growth. He estimated that differences in the levels of human capital between countries completely offset the difference in the rate of return on capital between poor and rich countries and posited that capital failed to flow equally between countries because of capital market imperfections (Lucas, 1990).

"The Marshall-Young-Romer Model"

Romer presented a production function with specialized inputs, which behave like an externality. The production function stated that

(36.) $Y_i = L_i^{1-a} \Sigma^{\infty}_{=1} x_i^{a}$, $i=1,2,3, \ldots, n, \quad a<1.$

where x_i stood for specialized intermediate inputs.[100] Each firm in a region produced one input and was monopolistically competitive. Together the firms produced a range M of inputs, that cost κ in terms of resources foregone. That is,

(37.) $\kappa = x_i M,$

or

$$\Sigma x_i = \kappa/M.$$

The production function could be rewritten as

(38.) $Y_i = L_i^{1-a}\kappa_i^{a}M^{a}.$

Throughout Romer's presentation of this model, the last term in the production function was written as M^{1-a}, which is incorrect. The mistake had no effect on the substance of the argument. Many readers (not even Romer in reading the proofs), taking Romer at his word, would not notice the error. If so, one might ask, why publish the mathematics at all? Following the mathematics enforced a distinct mode of thinking and argument. Thus the whole dispute between neoclassical and new classical growth theory came down to one about specifications and formalisms.

In the production function (equation 38), each firm planned the use of its paid inputs L_i, κ_i, on the assumption of constant returns to scale, without taking into account the range of inputs produced by the region. Yet, the greater the range of inputs, the greater the productivity of the region. That is, at the regional level, there were increasing returns to scale. As Romer detailed, "(i)f you wanted to set up a business to produce new computer chips, land in Nebraska would be cheaper [than in Silicon Valley], but just try to find a firm nearby with the right equipment for baking, etching, and testing silicon wafers" (1989, p.108). The extent of the market for specialization was limited by the costs of transport. This model generated a static pattern of trade based on geographical features, which replicated the persistent cross-country differences in growth and correlation between exports and growth.[101]

D. The Hamiltonian Formalisms

The new growth models applied the Pontryagin-Hamiltonian to demonstrate the existence of the (sub)optimal path of growth. Thus Lucas, for instance, "hop(ed) that this application of Pontryagin's

Maximum Principle, essentially taken from David Cass (1961), is familiar to most of you. I will be applying these same ideas repeatedly in what follows" (1988, p.9).[102] Romer stressed that "the central tool used in the characterization of dynamic competitive equilibrium models is a [continuous time extension] of the Kuhn-tucker theorem" and spent six pages reviewing the "Ramsey-Solow-Cass-Koopmans type" model (Romer, 1989, p.70).[103]

The new graduate economics textbooks train students to apply the Hamiltonian formalisms to growth. Blanchard and Fisher (1989) asked students several questions on this topic.

Blanchard and Fisher:
Chapter 2, Problems, Question 9

This question required students to read an unpublished paper by Rebelo (1987). How, the students were asked, did Rebelo "reproduce the Kaldor-Solow fact of growth"? The answer went as follows.

Assuming a perfectly competitive economy with intertermporal efficiency pricing, Rebelo set up the utility maximizing problem originally posed by Ramsey,

(39a.) $_{\max}J = \int_0^\infty U(C(t))e^{-\rho t}dt.$

In order to generate a constant rate of growth, Rebelo made the output a linear function of capital,

(39b.) $K = BKs \quad 0\leq s\leq 1,$

where B was the average product of capital (Y/K) and s was the propensity to save. The production of these consumer goods involved either constant or diminishing returns, that is,

(39c.) $C = A[(1-s)K]^{\alpha}, \quad 0 \leq \alpha \leq 1.$

Transforming the integral into a Hamiltonian gave

(40.) $H = U(C) + p_K BKs + p_C[A((1-s)K)^{\alpha} - C],$

where p_K, p_C were prices of capital and consumer goods in competitive markets with perfect foresight. The marginal conditions for an efficient economy were as follows:

(41a.) $p_C = U'[A((1-s)K]^{\alpha},$

that is, the price of consumer goods equalled the marginal utility of consumption;

(41b.) $p_K B = p_C \alpha[A((1-s)K]^{\alpha-1},$

which was the firm's profit-maximizing condition;[104]

(41c.) $\dot{p}_K = -\partial H/\partial K + \rho p_K,$

which was the first "canonical" equation;

(41d.) $\dot{K} = \partial H/\partial p_K = BKs,$

which was the second "canonical" equation; and

(41e.) $\lim_{t \to \infty} p_K K e^{-\rho t},$

which ensured a convergent solution. Given these five conditions, and the initial capital stock, Rebelo solved for the steady state growth rates of capital, consumption and output.

Rebelo contrasted his result with Solow's, stating that

> Solow (1956) concluded that the savings rate determines only the steady state levels of the

> different variables but not their growth rates ...Our simple model can be used to illustrate that this result is an artifact of the exogenous nature of steady state growth in the neoclassical model

(Rebelo, p.15). Of course, that growth depended on savings in Rebelo's model was merely an artifact of his use of the Hamiltonian formalisms.

Blanchard and Fisher,

Chapter 2, Problems, Question 8.

In this question Blanchard and Fisher asked students to solve for the optimal path of an economy with increasing returns to scale. The answer can be found in Romer, 1989.

Romer stated the traditional model of an externality,[105]

(42a.) $Y_i = F(K_i^a L_i^{1-a} K_\alpha^\eta)$, $a<1$, $\eta>0$, $(a+\eta)>1$, $i = 1,2,3 \ldots n$,

where Y_i, K_i, L_i stood for the output and inputs of the perfectly competitive firm i and K_α stood for the capital of the n-1 firms in the industry of which the firm i was a part.[106] The capital of the industry was a proxy for the state of knowledge, which increased when individual firms invested in research and development. Knowledge was a public good because it was nonrival (the same information could be produced at zero cost) and nonexcludable (patents did not give property rights in general knowledge). Since firms ignored the nonappropriable contribution to productivity, factors were paid according to their subjective marginal product.

Recent results of tests of the existence of externalities have been mixed.[107]

Romer next defined the planning problem, to maximize utility over the infinite horizon, given a constant rate of growth of the labor force,

(43a.) $_{\max}J = \int_0^\infty u(c(t))e^{-\rho t}dt.$

Maximization was subject to the accounting identity

(43b.) $\dot{k}(t) = f(k) - c(t).$

and the production function constraint

(42b.) $f(k) = k(t)^{a+n}K_r(t)^\eta, \qquad K_r = K_a/K,$

where K stood for the total capital stock ($K = k_i + K_a$), k was normalized for labor, and K_r represented the importance of the positive externality. Transforming the integral into the current valued Hamiltonian gave,

(44.) $H(k,p,K_r) = u(c) - p[(k^{\alpha+n}N^\eta)-c].$

In this expression, the multiplier p was the imputed price of the investment and the bracketed term after it stood for investment. The two "canonical" equations were

(45a.) $\dot{k} = \partial H/\partial p$

and

(45b.) $\dot{p}(t) = -\partial H/\partial k + \rho p,$

where the second term stood for the price of the marginal product of capital,

(45c.) $\partial H/\partial k = p\ aK_r^\eta k^{a+\eta-1} \equiv p\ f'(k).$

In steady state growth, the price of investment was constant, (45d.) $\dot{p}=0$.

This formulation yielded an intriguing result. In the case of diminishing returns to capital -- with $(a + \eta)<1$ -- as the stock of capital increased, the marginal product of capital tended to equal the rate of discount -- $f'(k)=\rho$ -- which was consistent with the "golden rule". As a result, output per capita would be constant, as in the Cass-Koopmans model. In the case of increasing returns to capital -- $(a + \eta)>1$ -- the marginal product of capital tended to exceed the rate of discount --$f'(k)>\rho$. In the phase plane, there would be no stationary point. The economy would experience a positive, constant rate of per capita growth with a rising capital-labor ratio, which was consistent with Kaldor's stylized facts.

In contrast to the neoclassical model, in which productivity growth was exogenous, Romer argued that in his model growth was endogenous, dependent as it was, on the externality that arises from investment in knowledge. Thus his model, unlike the neoclassical model, offered a basis upon which to analyze the effects of taxation. In reaching this enthusiastic conclusion, however, Romer confused his model with reality. His mathematical result, he stated,

> shows that increasing returns [to scale] are not by themselves enough to sustain persistent growth. In addition, the private marginal product [of capital] must not fall too rapidly as k grows

(1989, p.94). This restriction was merely an artifact that arose from the use of the Hamiltonian formalism, given the assumption of an external effect of investment in knowledge.

E. The Limitations of the Hamiltonian Formalisms

The contribution of the new growth models has been to put the old ideas of externalities and specialization into an acceptable formal framework through which to trace out their role in economic growth, so as to replace the neoclassical model. As Lucas stated,

> while it is not exactly wrong to describe these differences (in cross-country "knowledge") by an exogenous, exponential trend ..., neither is it useful to do so. We want a formalism that leads us to think about individual decisions to acquire knowledge, and about the consequences of these decisions for productivity

(1988, p.5).

Mathematics, according to Lucas, is no mere neutral language, as Samuelson has claimed. Rather, mathematics provides the problem-solving techniques that provides the tools to forge ahead in certain directions. Some mathematical formalisms have superior heuristic power. They let economists say more than other mathematical syntaxes. Each syntax also imposes its own restrictions (an issue that Lucas ignored) on what economists can and cannot say.

The formalisms of the new Walrasian growth theory admit an interpretation in terms of realistic, institutional detail about which the neoclassical model of growth had nothing to say. Within the Walrasian paradigm itself, the new growth theory offers an improvement over the stationary, capital theoretic models of the 1960s. Orthodox growth theory now has more general models, which can produce steady per capita growth.

At the same time, the Walrasian growth model, of a Hamiltonian function constrained by a production function, restricts economic thinking about growth to a utility maximizing framework that assumes a competitive economy in continuous equilibrium in which output depends solely on factors of supply. The mathematical conventions of the the new growth theory mean that the growth path of the economy into infinite time is entirely deterministic. It is impossible in this context to think of growth as a developmental, time-dependent process, with largely unpredictable global properties.

The Hamiltonian formulation in the new growth models is internally inconsistent. It implies that utility per capita is conserved along an optimal path along which consumption per capita is growing. The new classical economists specify production functions that yield increasing returns to public knowledge. But advances in the state of knowledge, which take time, are irreversible. And the Hamiltonian system produces an optimal path that is time-independent and irreversible. The new classical growth project finds itself hoisted upon its own petard.

Notes

1. This Working Paper is a draft of a chapter for a book on macroeconomics. The author thanks M. Deedy, P. Mirowski, R. W. Clower, and P. Albin for their comments.

2. O'Hear, p.183; Segrè, 1974, pp.242, 253.

3. Cohen, 1985, p.101.

4. Segrè, 1974, pp.155-56.

5. Segrè, 1976, p.222.

6. This section would not have been written without the contributions of Mirowski (1986, 1989a, 1989b, 1990).

7. Lanczos, 1949; Wilson, 1912.

8. Hamilton, 1834, p.251.

9. Hamilton's expression read T = U + H, where U was the "force function". In the 1840s, Helmholtz and others clearly stated the law of energy conservation. They defined potential energy, V, with V=-U. (Mirowski, 1989, p.34.)

10. Hamilton, 1834, p.248.

11. Hamilton, 1834, p.250.

12. Hamilton, 1834, pp.250, 251.

13. Hamilton, 1834, p.279.

14. Griffiths, 1985.

15. Wulwick, 1990.

16. pp. 478, 493

17. Weintraub, 1979.

18. Samuelson, 1983, pp. 317-18, 334ff.

19. Griffiths, 1984, pp.13-14.

20. Wolfson, 1972.

21. Emphases added by the author.

22. See equation 1, above.

23. Kalman, 1963, p.315.

24. Bellman, 1954, pp.82-83, 247.

25. Samuelson, 1983, p.363 n.1 and chapter 12; 1960, p.87 n.9; Samuelson and Solow, 1956, p.543 n.2.

26. 1983, p.8.

27. Dorfman et al, 1958, pp.266, 310-15.

28. 1983, p.8.

29. Burmeister and Dobell, 1970, pp.402-03.

30. Solow, 1956a.

31. Solow, 1956a, pp.79-81; Samuelson and Solow, 1956, p.1; Samuelson, 1963, pp.194, 201, 205.

32. See equation 12.

33. Samuelson (1947/1983) p. 173.

34. Samuelson, 1960, p.79 n2.

35. 1960, p.82.

36. p.554.

37. See equation 7a above.

38. See equation 7b, above.

39. pp.540, 556.

40. See above, equation 5a.

41. pp.558; 546, 554. See also, Intriligator, 1970, p.330, 335.

42. 1988, p.309.

43. (1948). An earlier essay appear in the Economic Journal, 1939.

44. Harrod, 1948, 63-100.

45. Swan, 1956, p.337 n.6 and figure 1.

46. Archibald and Lipsey, 1980, p.368.

47. Solow, 1956b, reprinted in Stigliz and Uzawa, editors, 1969.

48. J. Robinson, 1933; Harrod, 1937.

49. 1988, 310; Economic Report of the President, 1962, pp.316, 335-41, 372.

50. 1959, p.98.

51. Abramowitz, 1956.

52. 1963, p. 193.

53. The condition of a steady state dated back to the von Neumann (1938) fixed-coefficient model. The English translation was von Neumann (1945-46).

54. Commodity flows are discounted by $n + \rho$ and utility flows by ρ, but utility per worker by ρ-n. The mathematical argument went as follows:

$$\begin{aligned} J(\rho) &= \int e^{-\rho t} LU(C/L)\,dt \\ &= \int e^{-\rho t} L_o e^{nt} U(C/L)\,dt \\ &= L_o \int e^{-(\rho-n)t} LU(c)\,dt, \qquad c = C/L. \\ &= L_o \int e^{-(\sigma)t} U(c)\,dt, \qquad \sigma = \rho - n. \end{aligned}$$

55. See note 57, below.

56. Griffiths, 1984, p.161. See equation 12, above.

57. This result obtains from the following:
By definition, $f(k) = f(k) - nk$.
Then, $f'(k) = f'(k) - n$.
The geometry gave $f'(k) = \rho$.
Thus $f'(k) = n + \rho$

58. Solow, 1970, p.84.

59. Albin, 1983, p.262.

60. Shell, 1988, p.589.

61. Pontryagin, L. S., et al (1959), "Optimal control processes", Uspeckhi Mat. Nauk, V.14, pp.3-20.

62. Kalman, 1963, p.314-315.

63. See equation 5a, above..

64. Kalman, 1963, p.320; see equations 15 and 16, above.

65. Uzawa, 1965, p. 21.
A draft of the Cass, paper appeared as: Cass, D. (1963), "Optimal Savings in an Aggregative Model of Capital Accumulation", Technical Report no.5, Institute for Mathematical Studies in Social Sciences, Stanford University.

66. Samuelson, 1947/83, pp.69-70 ff.

67. Bellman, 1965, p.263. See p.26, above.

68. Samuelson and Solow, 1956, pp. 547 n.3, 549 n.4; Samuelson, 1960, p.77.

69. See p.52, above.

70. Equations 7, 8 and 14, above.

71. Magill, 1970, pp.56-57.

72. Magill, 1970, pp. 31, 75. See equation 6, abobe.

73. As Cass stated, "the limiting quasi-stationary path is nothing more than the golden rule path", $f'(k) = n$ (p.236). Equations 29a-c are simpler than Cass's differential equation.

74. Massé, 1946, p.58; Dorfman, 1969, p.819.

75. See equations 29b, 29c.

76. Shell, 1969, p.246.

77. Burmeister and Dobell, 1970, Figures 46 and 48 and pp.372-73.

78. Kalman, 1963, p.315; Bellman, 1965, p.253; Magill, 1970, p.13.

79. Burmeister and Dobell, 1970, pp. 396, 398.

80. Studies in Alternative Research Programs, October 1989, Capri, Italy. Organized by M. Blaug and N. de Marchi or the Latsis Foundation. Paper by N. J. Wulwick.

81. Romer, 1986, p.1005, and n.2; 1989a, p.7.

82. Quandt, 1976.

83. Kaldor, 1957.

84. Kaldor and Mirrlees, 1962.

85. Kaldor, 1958/63.

86. pp.515, 516-520, 345, 363; ed. Whitaker, Vol. 2, p.184.

87. Blitch, 1983 and forthcoming.

88. Young, 1928, p.528.

89. Blitch, forthcoming.

90. Meade, 1950, p.152.

91. Blitch, forthcoming.

92. p.1.

93. Kaldor, 1958/1963, pp.178-179.

94. Kaldor, 1957, 1958/1963; Wulwick, 1989.

95. Hahn and Matthews, 1964.

96. Kaldor, 1958/1963, p.215 n.1.

97. Cornwall, 1976, 1977.

98. Krugman, 1985, p.45.

99. The Review of Economic Studies held a symposium on the optimal use of exhaustible resources in 1974.

100. Krugman, 1979, 1985; Helpman, 1984; Romer, 1986, 1989a

101. Helpman, 1984.

102. The Cass article in fact appeared in 1965.

103. Barro, 1989, p.6.

104. Rebelo, 1987, pp.11, 15.

105. Helpman, 1984.

106. Chipman, 1970, p.351; Helpman, 1984; Krugman, 1985, p.36; romer, 1989a, 1989b.

107. At an NBER conference on growth, October, 1989, two papers gave conflicting evidence on the existence of externalities. Benhabib and Jovanovic (1989) did not find evidence of a large positive externalities on the capital input using quarterly and annual post-war US data. Cabellero and Lyons (1989) found evidence of externalities for US manufacturing industries, 1953-80.

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